Orbital stability of traveling waves for the one-dimensional Gross-Pitaevskii equation

Patrick Gérard† and Zhifei Zhang †‡
† Université Paris-Sud, Mathématiques, Bât. 425, 91405 Orsay Cedex, France.
E-mail: patrick.gerard@math.u-psud.fr

‡ School of Mathematical Science, Peking University, Beijing 100871, China
E-mail: zfzhang@math.pku.edu.cn

July 25, 2008

Abstract

In this paper, we prove the nonlinear orbital stability of the stationary traveling wave of the one-dimensional Gross-Pitaevskii equation by using Zakharov-Shabat's inverse scattering method.

1 Introduction

The Gross-Pitaevskii equation

$$\begin{cases} iu_t + \Delta u = (|u|^2 - 1)u, \quad x \in \mathbb{R}^d \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

which models the dynamics of Bose-Einstein condensates, superfluids, has received a lot of interest during the recent years. For a recent state of the art, we refer to the Proceedings [6] and to references therein. At least formally, (1.1) can be seen as a Hamiltonian evolution equation associated to the Ginzburg-Landau energy

$$H(u) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla_x u|^2 + \frac{1}{4}(|u|^2 - 1)^2 dx, \tag{1.2}$$

defined on the energy space

$$E = \{u \in H^1_{\rm loc}(\mathbb{R}^d) : \nabla_x u \in L^2(\mathbb{R}^d), |u|^2 - 1 \in L^2(\mathbb{R}^d)\}.$$

The unusual conditions at infinity imposed by the finiteness of the Ginzburg-Landau energy give rise to the existence of many traveling waves solutions to (1.1). In what follows, we shall restrict our purpose to space dimension $d = 1$, refering to [1], [2], [3] and references therein for the case $d = 2$ and $d = 3$. In the one-dimensional case, solutions of the form

$$u(t,x) = U(x - ct)$$

are completely characterized by solving an ordinary differential equation (see e.g. [3]). Besides constant solutions of modulus 1, which correspond to the solutions with $H(u) = 0$, these solutions are given by

$$U_c(x) = \sqrt{1 - \frac{c^2}{2}} \tanh\Big(\sqrt{1 - \frac{c^2}{2}} \frac{x}{\sqrt{2}}\Big) + i\frac{c}{\sqrt{2}} \ , \tag{1.3}$$

up to a multiplicative constant of modulus 1. An important question is then the stability of such objects for a natural distance on the energy space E, say

$$d_E(u, v) = |u(0) - v(0)| + \|u' - v'\|_{L^2(\mathbb{R})} + \||u|^2 - |v|^2\|_{L^2(\mathbb{R})} \ . \tag{1.4}$$

First of all, let us mention that the notion of stability in this case has to be properly defined, taking into account the existence of a continuum of traveling waves corresponding to a continuum of velocities. For example, using the formula (1.3), it is clear that $d_E(U_c, U_{c_0})$ tends to 0 as c tends to c_0, but if $c \neq c_0$, we have

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |U_c'(x - ct) - U_{c_0}(x - c_0t)|^2 dx = \|U_c'\|^2_{L^2} + \|U_{c_0}'\|^2_{L^2}.$$

For this reason, we shall say that U_c is orbitally stable for the distance d_E on E if, denoting by τ_y the translation operator

$$\tau_y f(x) = f(x - y),$$

we have, for every solution u of (1.1)

$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}} d_E(\tau_y u(t), U_c) \to 0, \quad \text{as} \quad d_E(u(0), U_c) \to 0.$$

In the case of the velocity $c \neq 0$, Lin [10] proved the orbital stability of U_c for (1.1) by using the Grillakis-Shatah-Strauss theory. His proof is based on the hydrodynamical form of (1.1): the solution is written as $u = (1 - r)^{\frac{1}{2}} e^{i\theta}$ so that the equations expressed in terms of new variables (r, θ_x) turn into a Hamiltonian system which fits in the framework of [9]. Then Lin reduces the orbital stability of U_c to the condition

$$\frac{d}{dc} P(U_c) < 0,$$

where the renormalized momentum $P(u)$ is defined by

$$P(u) = \int_{\mathbb{R}} \mathrm{Im}(\overline{u}u')\Big(1 - \frac{1}{|u|^2}\Big) dx.$$

This approach is valid for non-zero velocities c, because in this case U_c does not vanish on $\mathbb{R}$. A major difficulty in extending the above approach to the case of zero velocity is that $U_0(x)$ vanishes at some point so that $P(U_0)$ is not defined. In [5], Di Menza and Gallo proved the linear stability of U_0 under H^1 perturbations, and performed numerical computations which suggest the nonlinear orbital stability. A very recent result by Béthuel, Gravejat, Saut and Smets [4] proved a weak form of orbital stability of U_0 for d_E, allowing to renormalize the solution by factors of modulus 1. In the present paper, we prove that this renormalization by factors of modulus 1 is useless, at least for sufficiently smooth and decaying perturbations. Our main result is the following nonlinear orbital stability of U_0 for (1.1).

Theorem 1.1. *Assume that the initial datum of (1.1) has the form*

$$u_0(x) = U_0(x) + \varepsilon u_1(x), \qquad U_0(x) = \tanh(\frac{x}{\sqrt{2}}),$$

where $u_1(x)$ satisfies the following condition

$$\sup_{x\in\mathbb{R}} | < x >^4 \partial^k u_1(x)| \leq 1, \qquad for \quad k \leq 3. \tag{1.5}$$

Then if $\varepsilon > 0$ is small enough, there exists a unique solution $u(t,x)$ of (1.1) such that

$$\forall t \in \mathbb{R} \ , \ \exists y(t) \in \mathbb{R} \ , \ \|\tau_{y(t)} u(t,.) - U_0\|_{L^\infty} \leq C\varepsilon, \qquad for \quad 0 \leq t < +\infty. \tag{1.6}$$

Using a functional analytic argument from [4], Theorem 1.1 easily yields the orbital stability for d_E, at least for sufficiently smooth and decaying perturbations.

Corollary 1.1. *For every $\delta > 0$, there exists $\varepsilon > 0$ such that, if*

$$\sup_{x\in\mathbb{R}} | < x >^4 \partial^k u_1(x)| \leq 1, \qquad for \quad k \leq 3,$$

then the solution u of (1.1) satisfies

$$\forall t \in \mathbb{R} \ , \ \exists y(t) \in \mathbb{R} \ , \ d_E(\tau_{y(t)} u(t), U_0) \ \leq \delta.$$

More precisely, we shall prove that, in Theorem 1.1 and Corollary 1.1, one can choose $y(t) = \beta t$ and we shall give an interpretation of the parameter β. Our strategy for proving Theorem 1.1 follows the inverse scattering method as developed by Zakharov and Shabat in [11]. Recall that this method is based on the following observation. Let u be a function of (t,x). Denote by u^* the complex conjugate of u. Set

$$L_u = i \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \ , \tag{1.7}$$

and

$$B_u = -\sqrt{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} \frac{|u|^2-1}{\sqrt{3}+1} & iu_x^* \\ -iu_x & \frac{|u|^2-1}{\sqrt{3}-1} \end{pmatrix} \ . \tag{1.8}$$

It is easy to verify that

$$[L_u, B_u] = \begin{pmatrix} 0 & -u_{xx}^* + (|u|^2-1)u^* \\ u_{xx} - (|u|^2-1)u & 0 \end{pmatrix}.$$

Thus $u = u(t)$ satisfies the Gross-Pitaevskii equation (1.1) if and only if it satisfies the operator evolution equation

$$\frac{d}{dt} L_u = i[L_u, B_u]. \tag{1.9}$$

The above evolution equation implies that spectral properties of L_u are easily handled as t varies. Then the solution u at time t is recovered from spectral data of $L_{u(t)}$ from a system

of integral equations. We shall follow this procedure step by step in the perturbation context of Theorem 1.1 and deduce the parameter β from the spectral data of L_{u_0}.

This paper is organized as follows. In section 2, we classically discuss the properties of generalized eigenfunctions for L_u, if $u - U_0$ is sufficiently smooth and decaying at infinity, and we introduce a representation formula for these Jost solutions involving a kernel Ψ which will be the center of our analysis. In section 3 we discuss the properties of transition (or scattering) coefficients for L_u, in particular in the perturbation context of Theorem 1.1, while section 4 is devoted to the evolution of these coefficients deduced from equation (1.9). Section 5 is devoted to establishing the fundamental system of Marchenko equations which gives the kernel Ψ from the transition coefficients. In section 6, we use the system of Marchenko equations to prove further information about transition coefficients, in particular the fact that the transmission coefficient admits exactly one zero λ_0 if u is a smooth and decaying perturbation of U_0. Finally, Theorem 1.1 is proved in section 7, where it is shown that the translation vector $y(t)$ at time t in can be taken as $y(t) = -2\lambda_0 t$. Appendix A is devoted to the proof of two technical lemmas, while appendix B explains how to derive Corollary 1.1 from Theorem 1.1 and a compactness argument in [4].

Throughout this paper, z^* denotes the complex conjugate of the complex number z.

Acknowledgements. The first author would like to thank J.-C. Saut for suggesting this problem. This paper was written while the second author was visiting the Mathematics Department at Orsay as a postdoctoral fellow. He would like to thank the hospitality and support of the Department. The second author is partially supported by NSF of China under Grant 10601002.

2 Jost solutions and their properties

In this section, we assume that u is a C^4 function on $\mathbb{R}$ satisfying

$$\sup_{x\geq 0}\left|\frac{d^k}{dx^k}\left(u(x)-1\right)\right|\langle x\rangle^4+\sup_{x\leq 0}\left|\frac{d^k}{dx^k}\left(u(x)+1\right)\right|\langle x\rangle^4<+\infty\ ,\ 0\leq k\leq 3. \tag{2.1}$$

In view of the Lax pair framework recalled at the end of the introduction, the scattering problem associated with the Gross-Pitaevskii equation is

$$L_u\chi = E\chi,\quad \chi=\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix},\quad E\in\mathbb{R}\ , \tag{2.2}$$

where L_u is defined by (1.7). We make the change of variables

$$\chi_1=(\sqrt{3}-1)^{\frac12}e^{i\frac{Ex}{2}}v_1,\quad \chi_2=(\sqrt{3}+1)^{\frac12}e^{i\frac{Ex}{2}}v_2.$$

Then (2.2) is reduced to

$$\begin{cases} i\dfrac{\partial v_1}{\partial x}+q^*v_2=\lambda v_1,\\ -i\dfrac{\partial v_2}{\partial x}+qv_1=\lambda v_2,\end{cases} \tag{2.3}$$

where $\lambda = \frac{\sqrt{3}}{2}E,\ q = \frac{\sqrt{2}}{2}u$. Introducing the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ Q(x) = \begin{pmatrix} 0 & q^*(x) \\ q(x) & 0 \end{pmatrix} \ ,$$

system (2.3) reads

$$iM\partial_x v + Qv - \lambda v = 0 \ . \tag{2.4}$$

Notice that system (2.3) is invariant with respect to the involution

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \widetilde{v} = \begin{pmatrix} v_2^* \\ v_1^* \end{pmatrix} .$$

In other words, $\widetilde{v}$ is also a solution of (2.3) if v is a solution of (2.3). Moreover, if both v and w are solutions of (2.3), then their Wronskian defined by

$$\{v, w\} = v_1 w_2 - v_2 w_1$$

does not depend on x.

By (2.1), we find that $q \to \frac{\sqrt{2}}{2}$ as $x \to +\infty$, and $q \to -\frac{\sqrt{2}}{2}$ as $x \to -\infty$. Set

$$X_1^+ = e^{-i\zeta x} \begin{pmatrix} 1 \\ \sqrt{2}(\lambda - \zeta) \end{pmatrix}, \quad X_2^+ = e^{i\zeta x} \begin{pmatrix} \sqrt{2}(\lambda - \zeta) \\ 1 \end{pmatrix}. \tag{2.5}$$

which are solutions of (2.3) with $q = \frac{\sqrt{2}}{2}$. Here $\zeta \in \mathbb{R}$ and satisfies $\lambda^2 - \zeta^2 = \frac{1}{2}$. Hence X_1^+, X_2^+ and all the functions we are going to define are strictly speaking functions on the hyperbola

$$H = \{(\lambda, \zeta) \in \mathbb{R}^2 \mid \lambda^2 - \zeta^2 = \frac{1}{2}\}. \tag{2.6}$$

Notice that ζ is a coordinate on each of the branches $H_\pm = H \cap \{\pm\lambda > 0\}$ of H. In what follows, we will often make the traditional abuse of notation which consists in suppressing the dependence on ζ, so that these functions appear as double-valued functions of λ. Similarly, we set

$$X_1^- = e^{-i\zeta x} \begin{pmatrix} 1 \\ -\sqrt{2}(\lambda - \zeta) \end{pmatrix}, \quad X_2^- = e^{i\zeta x} \begin{pmatrix} -\sqrt{2}(\lambda - \zeta) \\ 1 \end{pmatrix}, \tag{2.7}$$

which are solutions of (2.3) with $q = -\frac{\sqrt{2}}{2}$.

The Jost solutions ψ_1 and ψ_2 are the solutions of the system (2.3) with the following asymptotic forms at infinity

$$\psi_1 \sim X_1^+, \qquad \psi_2 \sim X_2^+, \quad \text{as} \quad x \to +\infty.$$

Similarly,

$$\varphi_1 \sim X_1^-, \qquad \varphi_2 \sim X_2^-, \quad \text{as} \quad x \to -\infty.$$

Let us recall why these functions are well defined. We decompose $Q(x)$ as

$$Q(x) = Q^+ + R^+(x) \ , \ Q^+ = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \ , \ R^+(x) = \begin{pmatrix} 0 & q^*(x) - \frac{\sqrt{2}}{2} \\ q(x) - \frac{\sqrt{2}}{2} & 0 \end{pmatrix} \ .$$

Then, using (2.4), $\psi_1 = X_1^+ + V_1$ where V_1 satisfies

$$i\partial_x V_1 + M(Q^+ - \lambda)V_1 = -MR^+\psi_1 \ ,$$

with $V_1(x,\lambda) \to 0$ as $x \to +\infty$, hence, by the Duhamel formula,

$$\psi_1(x,\lambda) = X_1^+(x,\lambda) + \int_x^{+\infty} S(x,y,\lambda)\psi_1(y,\lambda)\, d\lambda \tag{2.8}$$

where the matrix S is given by

$$S(x,y,\lambda) = -ie^{i(x-y)M(Q^+-\lambda)}MR^+(y) \ . \tag{2.9}$$

Notice that, since $\lambda^2 \geq \frac{1}{2}$ on H, the spectrum of $M(Q^+ - \lambda)$ is real. In view of the decay of $R^+(x)$ provided by (2.1), S satisfies

$$\forall a \in \mathbb{R}, \int_a^{+\infty} \sup_{x\geq a} |S(x,y,\lambda)|\, dy < +\infty.$$

Consequently, the integral equation (2.8) is of Volterra type, hence admits a unique solution ψ_1. We argue similarly for the other Jost functions. Moreover, (2.1) also implies that

$$\forall a \in \mathbb{R}, \int_a^{+\infty} \sup_{x\geq a} |\partial_\lambda^j S(x,y,\lambda)|\, dy < +\infty, \ j \leq 2.$$

Consequently, we can state

Lemma 2.1. *The Jost functions $\psi_1, \psi_2, \varphi_1, \varphi_2$ are C^2 functions on the hyperbola H defined by (2.6).*

2.1 A representation formula

In what follows, we are going to study further properties of the Jost functions by establishing the following representation formula,

$$\psi_1(x,\lambda) = X_1^+(x,\lambda) - \int_x^{+\infty} \Psi(x,y)X_1^+(y,\lambda)dy, \tag{2.10}$$

where $\Psi(x,y)$ is a matrix-valued function. Substituting (2.10) into (2.4), we find that the matrix $\Psi(x,y)$ should satisfy the following linear system

$$iM\partial_x\Psi + i\partial_y\Psi M - \Psi Q^+ + Q(x)\Psi = 0$$

together with the boundary conditions

$$iM\Psi(x,x) - i\Psi(x,x)M + R^+(x) = 0 \ , \ \int_x^{+\infty} |\Psi(x,y)|\, dy \to 0 \text{ as } x \to +\infty.$$

A simple computation shows that this system is equivalent to

$$\begin{aligned}
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\begin{pmatrix}\Psi_{11}(x,y)\\ \Psi_{22}(x,y)\end{pmatrix} &= i\begin{pmatrix}-\frac{\sqrt{2}}{2} & q^*(x)\\ -q(x) & \frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}\Psi_{12}(x,y)\\ \Psi_{21}(x,y)\end{pmatrix},\\
\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\begin{pmatrix}\Psi_{12}(x,y)\\ \Psi_{21}(x,y)\end{pmatrix} &= i\begin{pmatrix}-\frac{\sqrt{2}}{2} & q^*(x)\\ -q(x) & \frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}\Psi_{11}(x,y)\\ \Psi_{22}(x,y)\end{pmatrix},
\end{aligned} \tag{2.11}$$

together with the boundary conditions

$$
\begin{aligned}
&\Psi_{12}^*(x,x) = \Psi_{21}(x,x) = -\frac{i}{2}(q(x) - \frac{\sqrt{2}}{2}),\\
&\int_x^{+\infty} |\Psi(x,y)|\, dy \to 0 \text{ as } x \to +\infty.
\end{aligned}
\tag{2.12}
$$

By the symmetry of the system (2.11), we find that

$$
\Psi_{11} = \Psi_{22}^*, \qquad \Psi_{12} = \Psi_{21}^*. \tag{2.13}
$$

From the invariance of the involution, we have

$$
X_2^+ = \widetilde{X}_1^+, \qquad \psi_2 = \widetilde{\psi}_1.
$$

Hence, we can obtain a similar representation for ψ_2

$$
\psi_2(x,\lambda) = X_2^+(x,\lambda) - \int_x^{+\infty} \Psi(x,y) X_2^+(y,\lambda) dy. \tag{2.14}
$$

Similarly, we can also obtain the following representation for φ_1 and φ_2

$$
\begin{aligned}
\varphi_1(x,\lambda) = X_1^-(x,\lambda) - \int_{-\infty}^x \Phi(x,y) X_1^-(y,\lambda) dy,\\
\varphi_2(x,\lambda) = X_2^-(x,\lambda) - \int_{-\infty}^x \Phi(x,y) X_2^-(y,\lambda) dy,
\end{aligned}
\tag{2.15}
$$

where the matrix $\Phi(x,y)$ satisfies the linear system

$$
\begin{aligned}
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \begin{pmatrix} \Phi_{11}(x,y) \\ \Phi_{22}(x,y) \end{pmatrix} = i \begin{pmatrix} \frac{\sqrt{2}}{2} & q^*(x) \\ -q(x) & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \Phi_{12}(x,y) \\ \Phi_{21}(x,y) \end{pmatrix},\\
\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \begin{pmatrix} \Phi_{12}(x,y) \\ \Phi_{21}(x,y) \end{pmatrix} = i \begin{pmatrix} \frac{\sqrt{2}}{2} & q^*(x) \\ -q(x) & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \Phi_{11}(x,y) \\ \Phi_{22}(x,y) \end{pmatrix},
\end{aligned}
\tag{2.16}
$$

together with the boundary conditions

$$
\begin{aligned}
&\Phi_{12}^*(x,x) = \Phi_{21}(x,x) = \frac{i}{2}(q(x) + \frac{\sqrt{2}}{2}),\\
&\int_{-\infty}^x |\Phi(x,y)|\, dy \to 0 \text{ as } x \to -\infty\ .
\end{aligned}
\tag{2.17}
$$

Let us show how to solve the system (2.11), (2.12). Introduce the nonnegative variable

$$
p = \frac{1}{2}(y - x) \geq 0
$$

and set

$$
V(x,p) = \begin{pmatrix} \Psi_{12}(x,x+2p) \\ \Psi_{21}(x,x+2p) \end{pmatrix},\ W(x,p) = \begin{pmatrix} \Psi_{11}(x,x+2p) \\ \Psi_{22}(x,x+2p) \end{pmatrix},\ B(x) = i \begin{pmatrix} -\frac{\sqrt{2}}{2} & q^*(x) \\ -q(x) & \frac{\sqrt{2}}{2} \end{pmatrix}
$$

Then system (2.11) reads

$$\partial_x W(x,p) = B(x)V(x,p) \ , \ \partial_p V(x,p) - \partial_x V(x,p) = -B(x)W(x,p) \ ,$$

and (2.12) reads

$$V(x,0) = V_0(x) = \frac{i}{2}\left(q(x) - \frac{\sqrt{2}}{2}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \ , \ \int_0^{+\infty} (|V(x,p)| + |W(x,p)|)\, dp \to 0 \text{ as } x \to +\infty \ .$$

Writing W from the first equation as

$$W(x,p) = -\int_x^{+\infty} B(x')V(x',p)\, dx' \ , \tag{2.18}$$

we finally obtain the integral equation for V,

$$V(x,p) = V_0(x+p) + \int_0^p \int_{x+p-p'}^{+\infty} B(x+p-p')B(x')V(x',p')\, dx'\, dp' \ . \tag{2.19}$$

Notice that

$$B(z)B(z') = -\begin{pmatrix} \frac{1}{2} - q(z')q^*(z) & \frac{\sqrt{2}}{2}(q^*(z) - q^*(z')) \\ \frac{\sqrt{2}}{2}(q(z) - q(z')) & \frac{1}{2} - q^*(z')q(z) \end{pmatrix} \tag{2.20}$$

so that, in view of (2.1), if $z' \geq z \geq -a$ for a positive number a, then

$$|B(z)B(z')| \leq C(a)\langle z + a\rangle^{-4} \ .$$

For every integer $N \geq 2$, denote by $\mathcal{V}_N$ the space of functions $V = V(x,p)$ on $\mathbb{R} \times \mathbb{R}_+$ such that, for every $a > 0$,

$$\sup_{x \geq -a} \langle x\rangle^N \int_0^{+\infty} |V(x,p)|\, dp < +\infty$$

For every $V \in \mathcal{V}_2$, we set

$$Kv(x,p) = \int_0^p \int_{x+p-p'}^{+\infty} B(x+p-p')B(x')V(x',p')\, dx'\, dp' \ .$$

We claim that $Kv \in \mathcal{V}_{N+1}$ if $V \in \mathcal{V}_N$. Indeed, if $x \geq -a$,

$$|Kv(x,p)| \leq C(a) \int_0^p \langle x + p - p' + a\rangle^{-4} \int_{x+p-p'}^{+\infty} |V(x',p')|\, dx'\, dp' \ ,$$

thus, denoting by $\|f\|$ the L^1 norm of $f = f(p)$ on $\mathbb{R}_+$, we have

$$\|Kv(x,.)\| \leq C(a) \int_0^{+\infty} \langle x + s + a\rangle^{-4} \int_{x+s}^{+\infty} \|V(x',.)\|\, dx'\, ds$$

therefore

$$\|Kv(x,.)\| \leq C(a)\langle x + a\rangle^{-3} \int_x^{+\infty} \|V(x',.)\|\, dx'.$$

In particular, if $\|V(x,.)\| \leq D(a)\langle x+a\rangle^{-N}$, then

$$\|KV(x,.)\| \leq \frac{C(a)D(a)}{N-1}\langle x+a\rangle^{-N-1} \ .$$

Starting from $V_0(x,p) = V_0(x+p)$, we conclude by an easy induction that

$$\|K^n V_0(x,.)\| \leq \frac{C(a)^{n+1}}{(n+1)!}\langle x+a\rangle^{-n-4}$$

which implies that the series $\sum_{n=0}^{+\infty}(-1)^n K^n V_0(x,.)$ converges in $L^1(\mathbb{R}_+)$ uniformly for $x \geq -a$, for all $a > 0$. Moreover, denoting by V the sum of this series, V solves (2.19) and $V \in \mathcal{V}_3$. Finally, coming back to (2.18), we have $W \in \mathcal{V}_2$. It is now a routine to estimate similarly the derivatives of order $k \leq 3$ of V, W with respect to x, p, and to show that they belong to $\mathcal{V}_3, \mathcal{V}_2$ respectively. For future reference, we set

$$\underline{\Psi}(x,p) = \Psi(x, x+2p) \ ,$$

and we sum up the above results by

$$\forall a > 0, \forall x \geq -a, \ \int_0^{+\infty} |\partial^\alpha \underline{\Psi}(x,p)|\, dp \leq C(a)\langle x+a\rangle^{-3} \ , |\alpha| \leq 3. \tag{2.21}$$

2.2 Analytic continuations and a priori bounds

We are now in position to study the properties of Jost solutions ψ_1 and ψ_2. Introduce the Riemann surface

$$\Gamma = \{(\lambda,\zeta) \in \mathbb{C}\times\mathbb{C} \mid \lambda^2 - \zeta^2 = \frac{1}{2}\} \ ,$$

and denote by $\Gamma^\pm$ the two sheets of Γ corresponding to $\pm\mathrm{Im}\zeta > 0$. Notice that, ζ is a single-valued holomorphic function of λ on Γ^+ and on Γ^-. Also notice that

$$\overline{\Gamma}^\pm = \Gamma^\pm \cup H \ .$$

Lemma 2.2. *(1) The Jost solution $\psi_1(x,\lambda) \in C^3(\mathbb{R}_x)$ can be extended analytically to the lower sheet Γ^- and has the form*

$$\psi_1(x,\lambda) = X_1^+(x,\lambda) + \Psi_1(x,\zeta)X_1^+(x,\lambda).$$

(2) The Jost solution $\psi_2(x,\lambda) \in C^3(\mathbb{R}_x)$ can be extended analytically to the upper sheet Γ^+ and has the form

$$\psi_2(x,\lambda) = X_2^+(x,\lambda) + \Psi_2(x,\zeta)X_2^+(x,\lambda).$$

Here $\Psi_1(x,\zeta)$ and $\Psi_2(x,\zeta)$ satisfy

$$|\Psi_1(x,\zeta)| + |\Psi_2(x,\zeta)| \leq C(a)\langle\zeta\rangle^{-1}\langle x+a\rangle^{-3}, \quad \textit{for } x \geq -a, a > 0. \tag{2.22}$$

Proof. We rewrite (2.10) as

$$\begin{aligned}\psi_1(x,\lambda) &= X_1^+(x,\lambda) - \int_x^{+\infty} \Psi(x,y) X_1^+(y,\lambda) dy \\ &= X_1^+(x,\lambda) - 2\int_0^{+\infty} \underline{\Psi}(x,p) e^{-2i\zeta p}\, dp\, X_1^+(x,\lambda) \\ &\triangleq X_1^+(x,\lambda) + \Psi_1(x,\zeta) X_1^+(x,\lambda).\end{aligned}$$

We get by integrating by parts that

$$\Psi_1(x,\zeta) = \frac{i}{\zeta}\underline{\Psi}(x,0) + \frac{i}{\zeta}\int_0^{+\infty} \partial_p \underline{\Psi}(x,p) e^{-2i\zeta p} dp \ ,$$

which, in view of the estimates (2.21) on $\underline{\Psi}$, implies (2.22). Since (2) can be proved in a similar way, we omit its proof. □

Similarly, we state the following properties for the Jost solutions φ_1 and φ_2.

Lemma 2.3. *(1) The Jost solution $\varphi_1(x,\lambda) \in C^3(\mathbb{R}_x)$ can be extended analytically to the upper sheet Γ^+ and has the form*

$$\varphi_1(x,\lambda) = X_2^-(x,\lambda) + \Phi_1(x,\zeta) X_1^-(x,\lambda).$$

(2) The Jost solution $\varphi_2(x,\lambda) \in C^3(\mathbb{R}_x)$ can be extended analytically to the lower sheet Γ^- and has the form

$$\varphi_2(x,\lambda) = X_2^-(x,\lambda) + \Phi_2(x,\zeta) X_2^-(x,\lambda).$$

Here $\Phi_1(x,\zeta)$ and $\Phi_2(x,\zeta)$ satisfy

$$|\Phi_1(x,\zeta)| + |\Phi_2(x,\zeta)| \le C(a)\langle\zeta\rangle^{-1}\langle x-a\rangle^{-3}, \quad \textit{for } x \le a, a > 0. \tag{2.23}$$

2.3 The unperturbed case

As a next step we apply the above results to the particular case

$$q^0(x) = \frac{\sqrt{2}}{2}\tanh\left(\frac{x}{\sqrt{2}}\right) \ .$$

In this case, since q^0 is real valued, the kernel of equation (2.19) satisfies, in view of (2.20),

$$\begin{aligned}B(z)B(z')\begin{pmatrix}1\\-1\end{pmatrix} &= -\left(\tfrac{1}{2} - q(z)q(z') - \tfrac{\sqrt{2}}{2}q(z) + \tfrac{\sqrt{2}}{2}q(z')\right)\begin{pmatrix}1\\-1\end{pmatrix} \\ &= -\tfrac{1}{2}\left(1-\tanh\left(\tfrac{z}{\sqrt{2}}\right)\right)\left(1+\tanh\left(\tfrac{z'}{\sqrt{2}}\right)\right)\begin{pmatrix}1\\-1\end{pmatrix} \ ,\end{aligned}$$

so that $V_2(x,p) = -V_1(x,p)$ and (2.19) reads

$$\begin{aligned}V_1(x,p) = &-\frac{i}{2\sqrt{2}}\left(1-\tanh\left(\frac{x+p}{\sqrt{2}}\right)\right) \\ &-\frac{1}{2}\int_0^p\int_{x+p-p'}^{+\infty}\left(1-\tanh\left(\frac{x+p-p'}{\sqrt{2}}\right)\right)\left(1+\tanh\left(\frac{x'}{\sqrt{2}}\right)\right) V_1(x',p')\, dx'\, dp' \ .\end{aligned}$$

Introducing the following new variables

$$s = e^{\sqrt{2}(x+p)} > 0 \ , \ r = e^{\sqrt{2}p} \geq 1 \ , \ v(s,r) = V_1(x,p) \ ,$$

we obtain

$$v(s,r) = -\frac{i}{2\sqrt{2}(s+1)} - \int_1^r \int_s^{+\infty} \frac{v(s',r')}{(s+r')(s'+r')} \, ds' \, dr'$$

and it is easy to see that this integral equation admits for solution

$$v(s,r) = -\frac{i}{2\sqrt{2}(s+r)} \ .$$

Coming back to the original variables, we infer

$$\begin{aligned} \Psi^0_{12}(x, x+2p) = V^0_1(x,p) &= -\frac{ie^{-\sqrt{2}p}}{\sqrt{2}(1+e^{\sqrt{2}x})} \ , \\ \Psi^0_{11}(x, x+2p) = W^0_1(x,p) &= \frac{e^{-\sqrt{2}p}}{\sqrt{2}(1+e^{\sqrt{2}x})} \ . \end{aligned} \tag{2.24}$$

Through the representation derived in Section 2.1, we obtain

$$\begin{aligned} \psi^0_1(x,\lambda) &= e^{-i\zeta x} \begin{pmatrix} 1 - \frac{1}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2} - i(\lambda-\zeta)}{\frac{\sqrt{2}}{2}+i\zeta} \\ \sqrt{2}(\lambda-\zeta) - \frac{1}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2}i + (\lambda-\zeta)}{\frac{\sqrt{2}}{2}+i\zeta} \end{pmatrix} , \\ \psi^0_2(x,\lambda) &= e^{i\zeta x} \begin{pmatrix} \sqrt{2}(\lambda-\zeta) - \frac{1}{1+e^{\sqrt{2}x}} \frac{-\frac{\sqrt{2}}{2}i + (\lambda-\zeta)}{\frac{\sqrt{2}}{2}-i\zeta} \\ 1 - \frac{1}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2} + i(\lambda-\zeta)}{\frac{\sqrt{2}}{2}-i\zeta} \end{pmatrix} , \end{aligned} \tag{2.25}$$

and similarly, through Lemma 2.3,

$$\begin{aligned} \varphi^0_1(x,\lambda) &= e^{-i\zeta x} \begin{pmatrix} 1 - \frac{e^{\sqrt{2}x}}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2} + i(\lambda-\zeta)}{\frac{\sqrt{2}}{2}-i\zeta} \\ -\sqrt{2}(\lambda-\zeta) + \frac{e^{\sqrt{2}x}}{1+e^{\sqrt{2}x}} \frac{-\frac{\sqrt{2}}{2}i + (\lambda-\zeta)}{\frac{\sqrt{2}}{2}-i\zeta} \end{pmatrix} , \\ \varphi^0_2(x,\lambda) &= e^{i\zeta x} \begin{pmatrix} -\sqrt{2}(\lambda-\zeta) + \frac{e^{\sqrt{2}x}}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2}i + (\lambda-\zeta)}{\frac{\sqrt{2}}{2}+i\zeta} \\ 1 - \frac{e^{\sqrt{2}x}}{1+e^{\sqrt{2}x}} \frac{\frac{\sqrt{2}}{2} - i(\lambda-\zeta)}{\frac{\sqrt{2}}{2}+i\zeta} \end{pmatrix} . \end{aligned} \tag{2.26}$$

2.4 Perturbation analysis

We close this section by describing how the results of the previous subsections can be precised when q is assumed to be a perturbation of q^0 in the sense of Theorem 1.1, namely if

$$q(x) = q^0(x) + \varepsilon q^1(x) \ , \ \langle x \rangle^4 \left| \frac{d^k q^1}{dx^k}(x) \right| \leq 1 \ , \ 0 \leq k \leq 3 \ , \tag{2.27}$$

and ε is a small positive parameter. Revisiting the analysis of subsections 2.1 and 2.2, and comparing to the results of subsection 2.3, the following perturbation results can be easily proved.

Lemma 2.4. *The matrix-valued functions Ψ, Φ defined by (2.11), (2.12), (2.16), (2.17) can be written as*

$$\Psi(x,y) = \Psi^0(x,y) + \varepsilon\Psi^1(x,y) \ , \ \Phi(x,y) = \Phi^0(x,y) + \varepsilon\Phi^1(x,y) \ ,$$

with the following estimates : if we set, for $p \geq 0$,

$$\underline{\Psi}^1(x,p) := \Psi^1(x, x+2p) \ , \ \underline{\Phi}^1(x,p) = \Phi^1(x, x-2p) \ ,$$

then

$$\forall a > 0, \forall x \geq -a, \ \int_0^{+\infty} |\partial^\alpha \underline{\Psi}^1(x,p)| \, dp \leq C(a)\langle x+a\rangle^{-3} \ , |\alpha| \leq 3 \ , \tag{2.28}$$

$$\forall a > 0, \forall x \leq a, \ \int_0^{+\infty} |\partial^\alpha \underline{\Phi}^1(x,p)| \, dp \leq C(a)\langle x-a\rangle^{-3} \ , |\alpha| \leq 3 \ . \tag{2.29}$$

Lemma 2.5. *For fixed q^1, the Jost functions are real analytic of the paramter ε in a neighborhood of 0 in $\mathbb{R}$. Moreover, for ε small enough, they can be written as*

$$\psi_1(x,\lambda) = \psi_1^0(x,\lambda) + \varepsilon\psi_1^1(x,\lambda) \ , \ \psi_2(x,\lambda) = \psi_2^0(x,\lambda) + \varepsilon\psi_2^1(x,\lambda)$$
$$\varphi_1(x,\lambda) = \varphi_1^0(x,\lambda) + \varepsilon\varphi_1^1(x,\lambda) \ , \ \varphi_2(x,\lambda) = \varphi_2^0(x,\lambda) + \varepsilon\varphi_2^1(x,\lambda),$$

with

$$|\psi_1^1(x,\lambda)| + |\psi_2^1(x,\lambda)| \leq C\langle\zeta\rangle^{-1}\langle x\rangle^{-3}, \quad \textit{for } x \geq 0$$
$$|\varphi_1^1(x,\lambda)| + |\varphi_2^1(x,\lambda)| \leq C\langle\zeta\rangle^{-1}\langle x\rangle^{-3}, \quad \textit{for } x \leq 0,$$

and $\lambda \in \Gamma^-$ in the cases of ψ_1, φ_2, while $\lambda \in \Gamma^+$ in the cases of ψ_2, φ_1.

3 Transition coefficients and their properties

Now let ζ be real. Let $\psi_1(x,\lambda), \psi_2(x,\lambda), \varphi_1(x,\lambda)$, and $\varphi_2(x,\lambda)$ be the Jost solutions constructed in Section 2. Since their Wronskian is independent of x, we get by their asymptotic behaviour at infinity that

$$\{\psi_1, \psi_2\} = \{\varphi_1, \varphi_2\} = 4\zeta(\lambda - \zeta). \tag{3.1}$$

Thus, ψ_1 and ψ_2 are linearly independent for $\zeta \neq 0$. Hence, we can expand φ_1 and φ_2 uniquely as

$$\varphi_1 = a\psi_1 + b\psi_2, \qquad \varphi_2 = a^*\psi_2 + b^*\psi_1, \tag{3.2}$$

from which and (3.1), we get

$$a(\lambda) = \frac{\{\varphi_1, \psi_2\}}{4\zeta(\lambda - \zeta)}, \quad b(\lambda) = -\frac{\{\varphi_1, \psi_1\}}{4\zeta(\lambda - \zeta)}, \tag{3.3}$$

which are called the transition coefficients. In the case when $q(x) = \frac{\sqrt{2}}{2}\tanh(\frac{x}{\sqrt{2}})$, it is easy to compute by (2.25) and (2.26) that the corresponding transition coefficients denoted by $a^0(\lambda)$ and $b^0(\lambda)$ are

$$a^0(\lambda) = \frac{\lambda + \zeta - \frac{\sqrt{2}}{2}i}{\lambda + \zeta + \frac{\sqrt{2}}{2}i}, \quad b^0(\lambda) = 0.$$

Note that

$$\{\varphi_1, \varphi_2\} = (|a|^2 - |b|^2)\{\psi_1, \psi_2\},$$

and $\{\varphi_1, \varphi_2\} = \{\psi_1, \psi_2\}$, we have the normalization relation

$$|a(\lambda)|^2 - |b(\lambda)|^2 = 1, \quad \text{for } \lambda \in \mathbb{R},\, |\lambda| > \frac{\sqrt{2}}{2}. \tag{3.4}$$

We find by Lemma 2.2-2.3 and the definition of $a(\lambda)$ that the function $a(\lambda)$ can be extended analytically to the upper sheet Γ^+ . In what follows, we will study the properties about the transition coefficients. Let us begin with the following simple fact about the function $\zeta(\lambda)$ which will be constantly used: For $\lambda \in \mathbb{R}, |\lambda| \gg 1$

$$\zeta = \lambda + O(\frac{1}{\lambda}), \text{ for } \lambda > 0, \qquad \zeta = -\lambda + O(\frac{1}{\lambda}), \text{ for } \lambda < 0. \tag{3.5}$$

Lemma 3.1. *Let $a(\lambda), b(\lambda)$ be given by (3.3). Then there hold for $\lambda \in \mathbb{R}, |\lambda| > \frac{\sqrt{2}}{2}$*

$$|a(\lambda) - a^0(\lambda)| \le C\varepsilon|\zeta|^{-1}, \tag{3.6}$$

$$|b(\lambda)| \le C\varepsilon \min(|\zeta|^{-1}, |\zeta|^{-3}), \tag{3.7}$$

$$\left|\frac{b(\lambda)}{a(\lambda)}\right| \le \min(1, C\varepsilon|\zeta|^{-1}, C\varepsilon|\zeta|^{-3}). \tag{3.8}$$

Proof. We write

$$\psi_2 = \psi_2^0 + \varepsilon\psi_2^1, \quad \varphi_1 = \varphi_1^0 + \varepsilon\varphi_1^1\ . \tag{3.9}$$

Substituting (3.9) into the first formula of (3.3), we get

$$\begin{aligned} a(\lambda) &= \frac{\{\varphi_1, \psi_2\}}{4\zeta(\lambda - \zeta)} = \frac{\{\varphi_1^0 + \varepsilon\varphi_1^1, \psi_2^0 + \varepsilon\psi_2^1\}}{4\zeta(\lambda - \zeta)} \\ &= \frac{\{\varphi_1^0, \psi_2^0\}}{4\zeta(\lambda - \zeta)} + \varepsilon\frac{\{\varphi_1^0, \psi_2^1\} + \{\varphi_1^1, \psi_2^0\} + \varepsilon\{\varphi_1^1, \psi_2^1\}}{4\zeta(\lambda - \zeta)} \\ &\triangleq a^0(\lambda) + \varepsilon a^1(\lambda), \end{aligned}$$

which together with Lemma 2.5 and (3.5) gives (3.6). We next prove (3.7). Note that $\{\varphi_1^0, \psi_1^0\} = 0$, we get

$$\{\varphi_1, \psi_1\} = \varepsilon\{\varphi_1^0, \psi_1^1\} + \varepsilon\{\varphi_1^1, \psi_1^0\} + \varepsilon^2\{\varphi_1^1, \psi_1^1\}. \tag{3.10}$$

Thus, we can get by using Lemma 2.5 and (3.5) that

$$|b(\lambda)| \le C\varepsilon|\zeta|^{-1}. \tag{3.11}$$

In order to obtain the better decay estimate for $b(\lambda)$, we need to use the following more subtle argument. Multiplying by $e^{2i\zeta x}$ on both sides of (3.10) and taking the derivative with respect to x to the resulting equation, we get by using the fact that $\{\varphi_1, \psi_1\}$ is independent of x that

$$\begin{aligned} -8i\zeta^3 e^{2i\zeta x}\{\varphi_1, \psi_1\} = \varepsilon\big(\frac{d}{dx}\big)^3\big(&\{e^{i\zeta x}\varphi_1^0, e^{i\zeta x}\psi_1^1\} + \{e^{i\zeta x}\varphi_1^1, e^{i\zeta x}\psi_1^0\} + \\ &+\varepsilon\{e^{i\zeta x}\varphi_1^1, e^{i\zeta x}\psi_1^1\}\big). \end{aligned} \tag{3.12}$$

It is easy to verify by (2.26) that for $k \in \mathbb{N}$

$$|(\frac{d}{dx})^k (e^{i\zeta x}\varphi_1^0(x,\lambda))| \le C. \tag{3.13}$$

By the proof of Lemma 2.2 adapted to Lemma 2.5, we have

$$|(\frac{d}{dx})^k (e^{i\zeta x}\psi_1^1(x,\zeta))| \le C, \quad \text{for} \quad k \le 3, x \ge 0. \tag{3.14}$$

By summing up (3.13) and (3.14), we obtain

$$|(\frac{d}{dx})^3 \{e^{i\zeta x}\varphi_1^0, e^{i\zeta x}\psi_1^1\}| \le C\ .$$

Similarly, it can be proved that the other two terms on the right side of (3.12) are bounded by C . Thus, we get by using (3.5) and (3.12) that

$$|b(\lambda)| \le C\varepsilon|\zeta|^{-3}, \qquad for \quad |\zeta| \gtrsim 1,$$

which together with (3.11) gives (3.7). Now we turn to prove (3.8). We get by (3.4) that

$$\left|\frac{b(\lambda)}{a(\lambda)}\right| \le 1, \qquad |a(\lambda)| \ge 1,$$

from which and (3.7), it follows (3.8). ☐

Lemma 3.2. *The function $a(\lambda)$ has at most one zero. Assume that λ_0 is a zero of $a(\lambda)$, one has*
(1) $\lambda_0 \in (-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2})$ is simple;
(2) there exists a constant b_0 such that $\varphi_1(x,\lambda_0) = b_0\psi_2(x,\lambda_0)$ for any $x \in \mathbb{R}$;
(3) for ε small enough, there holds

$$|\lambda_0| \le C\varepsilon, \quad |b_0 - i| \le C\varepsilon, \quad |\mu_0 + 2| \le C\varepsilon, \tag{3.15}$$

where

$$\mu_0 = \frac{b_0}{\nu_0 a'(\lambda_0)}, \quad \nu_0 = (\frac{1}{2} - \lambda_0^2)^{\frac{1}{2}}.$$

Proof. Let us assume that λ_0 is a zero of $a(\lambda)$. From (3.4), we see that λ does not belong to $\{\lambda \in \mathbb{R}, |\lambda| > \frac{\sqrt{2}}{2}\}$, hence $Im\zeta > 0$.

Since

$$\{\varphi_1, \psi_2\}(\lambda_0) = 0,$$

there exists a constant b_0 such that

$$\varphi_1(x,\lambda_0) = b_0\psi_2(x,\lambda_0), \quad x \in \mathbb{R},$$

which implies that the system (2.3) with $\lambda = \lambda_0$ has a global L^2 solution on $\mathbb{R}$. Thus, the zeros of $a(\lambda)$ correspond to the eigenvalues of the system (2.3). From the self-adjoint character of

the system , it follows that λ_0 must lie on the segment $(-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2})$ of the real axis. We next prove that λ_0 is simple. It suffices to prove that $a'(\lambda_0) \neq 0$. By (3.3), we have

$$a'(\lambda_0) = \frac{\{\frac{\partial}{\partial\lambda}\varphi_1, \psi_2\}(x, \lambda_0) + \{\varphi_1, \frac{\partial}{\partial\lambda}\psi_2\}(x, \lambda_0)}{4\zeta_0(\lambda_0 - \zeta_0)}, \quad \zeta_0 = (\lambda_0^2 - \frac{1}{2})^{\frac{1}{2}} = i\nu_0 \ .$$

Set $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we get by using the equations (2.3) that

$$\begin{aligned} \frac{\partial}{\partial x}\{\frac{\partial}{\partial\lambda}\varphi_1, \psi_2\}(x, \lambda) &= -i\{\sigma\varphi_1, \psi_2\}(x, \lambda), \\ \frac{\partial}{\partial x}\{\varphi_1, \frac{\partial}{\partial\lambda}\psi_2\}(x, \lambda) &= i\{\sigma\varphi_1, \psi_2\}(x, \lambda), \end{aligned}$$

from which and the fact that the Jost solutions $\varphi_1(x, \lambda_0)$ and $\psi_2(x, \lambda_0)$ decay to zero as $|x| \to \infty$, it follows that

$$\begin{aligned} \{\frac{\partial}{\partial\lambda}\varphi_1, \psi_2\}(x, \lambda_0) &= -ib_0 \int_{-\infty}^{x} \{\sigma\psi_2, \psi_2\}(x', \lambda_0)dx', \\ \{\varphi_1, \frac{\partial}{\partial\lambda}\psi_2\}(x, \lambda_0) &= -ib_0 \int_{x}^{+\infty} \{\sigma\psi_2, \psi_2\}(x', \lambda_0)dx'. \end{aligned}$$

From the invariance of the involution of (2.3) and the uniqueness of the Jost solutions , we find that for $\lambda \in (-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2})$

$$\sqrt{2}(\lambda + \zeta)\psi_2(x, \lambda) = \widetilde{\psi}_2(x, \lambda),$$

which implies that

$$\{\sigma\psi_2, \psi_2\}(x, \lambda) = |\psi_2(x, \lambda)|^2.$$

Putting together the above formulas, we get

$$a'(\lambda_0) = -\frac{ib_0 \int_{-\infty}^{+\infty} |\psi_2(x, \lambda_0)|^2 dx}{2\sqrt{2}\zeta_0} \neq 0. \tag{3.16}$$

We now prove (3.15). Using the similar proof as in (3.6), we have

$$|a(\lambda) - a^0(\lambda)| \leq C\varepsilon|\zeta|^{-1}, \quad \text{for } \lambda \in (-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2}).$$

Set $\zeta = i(\frac{1}{2} - \lambda^2)^{\frac{1}{2}} = i\nu$ for $\lambda \in (-\frac{\sqrt{2}}{2}, +\frac{\sqrt{2}}{2})$. Then we have

$$|a^0(\lambda)| = \frac{\sqrt{2}|\lambda|}{1 + \sqrt{2}\nu} \geq \frac{\sqrt{2}}{2}|\lambda|.$$

Hence, we have at the zero point λ_0 that

$$|\lambda_0|^2|\zeta_0|^2 \leq C\varepsilon^2, \quad \zeta_0 = (\lambda^2 - \frac{1}{2})^{\frac{1}{2}}$$

which is equivalent to the inequality

$$\lambda_0^4 - \frac{1}{2}\lambda_0^2 + C\varepsilon^2 \geq 0.$$

We get by solving the above inequality that

$$\lambda_0^2 \in \Big[0, \frac{1}{4} - \frac{1}{2}\sqrt{\frac{1}{4} - 4C\varepsilon^2}\Big] \cup \Big[\frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} - 4C\varepsilon^2}, \frac{1}{2}\Big). \tag{3.17}$$

Note that $\frac{1}{4} + \frac{1}{2}\sqrt{\frac{1}{4} - 4C\varepsilon^2} \to \frac{1}{2}$ as $\varepsilon \to 0$, while by (3.4)

$$\liminf_{\lambda \to \pm\frac{1}{2}} |a(\lambda)| \geq 1,$$

which together with (3.17) implies that for ε small enough

$$|\lambda_0| \leq c\varepsilon. \tag{3.18}$$

Using the fact that $\varphi_1^0(x,0) = i\psi_2^0(x,0)$, we get

$$(i - b_0)\psi_2^0(x,0) = \varepsilon(b_0\psi_2^1(x,\lambda_0) - \varphi_1^1(x,\lambda_0)) - (\varphi_1^0(x,\lambda_0) - \varphi_1^0(x,0))$$
$$+ b_0(\psi_2^0(x,\lambda_0) - \psi_2^0(x,0)),$$

which together with Lemma 2.5 and (3.18) gives

$$|b_0 - i||\psi_2^0(0,0)| = \frac{\sqrt{2}}{2}|b_0 - i| \leq C\varepsilon|b_0 - i| + C\varepsilon,$$

which implies the second inequality of (3.15). We next prove the third inequality of (3.15). By the first two inequalities of (3.15) and (3.16), it suffices to prove that

$$\Big|\int_{-\infty}^{+\infty} |\psi_2(x,\lambda_0)|^2 dx - \sqrt{2}\Big| \leq C\varepsilon. \tag{3.19}$$

We write

$$\int_{-\infty}^{+\infty} |\psi_2(x,\lambda_0)|^2 dx = \int_0^{+\infty} |\psi_2^0(x,\lambda_0) + \varepsilon\psi_2^1(x,\lambda_0)|^2 dx$$
$$+ \frac{1}{|b_0|^2}\int_{-\infty}^0 |\varphi_1^0(x,\lambda_0) + \varepsilon\varphi_1^1(x,\lambda_0)|^2 dx,$$

from which and Lemma 2.5, it follows that

$$\Big|\int_{-\infty}^{+\infty} |\psi_2(x,\lambda_0)|^2 dx - \int_0^{+\infty} |\psi_2^0(x,\lambda_0)|^2 dx - \frac{1}{|b_0|^2}\int_{-\infty}^0 |\varphi_1^0(x,\lambda_0)|^2 dx\Big| \leq C\varepsilon. \tag{3.20}$$

On the other hand, from the exact formula (2.25-2.26) and (3.18), it is easy to verify that

$$\varphi_1^0(x,0) = i\psi_2^0(x,0), \quad \int_{-\infty}^{+\infty} |\psi_2^0(x,0)|^2 dx = \sqrt{2}, \quad \text{and}$$
$$\int_0^{+\infty} |\psi_2^0(x,\lambda_0) - \psi_2^0(x,0)|^2 dx + \int_{-\infty}^0 |\varphi_1^0(x,\lambda_0) - \varphi_1^0(x,0)|^2 dx \leq C\varepsilon,$$

which together with (3.20) imply (3.19).

Finally, we prove that $a(\lambda)$ has at most one zero. Assume that λ_0' is another zero of $a(\lambda)$. Then $|\lambda_0'| \le C\varepsilon$ by the above proof. We get by an exact computation that

$$a^0(\lambda_0) - a^0(\lambda_0') = \frac{i\sqrt{2}}{(1+\sqrt{2}\nu_0)(1+\sqrt{2}\nu_0')}\Big((1+\sqrt{2}\nu_0)(\lambda_0'-\lambda_0) + \sqrt{2}\lambda_0(\nu_0-\nu_0')\Big).$$

Hence, for ε small enough there exists a constant $c > 0$ such that

$$|a^0(\lambda_0) - a^0(\lambda_0')| \ge c|\lambda_0 - \lambda_0'|. \tag{3.21}$$

Using Lemma 2.5 and the fact that $|\lambda_0|, |\lambda_0'| \le C\varepsilon$, we have

$$|a^1(\lambda_0) - a^1(\lambda_0')| \le C|\lambda_0 - \lambda_0'|,$$

which together with (3.21) implies that $\lambda_0 = \lambda_0'$. The proof of Lemma 3.2 is complete. $\square$

4 The evolution of the transition coefficients

In this section, we will derive the evolution equations of the transition coefficients. Let us begin by deriving the evolution equations of the Jost solutions.

Differentiating (2.2) with respect to t, we get

$$\frac{\partial}{\partial t}L_u\chi + L_u\frac{\partial\chi}{\partial t} = E\chi,$$

from which and (1.9), it follows that

$$L_u(\frac{\partial\chi}{\partial t} + iB_u\chi) = E(\frac{\partial\chi}{\partial t} + iB_u\chi),$$

which together with the asymptotic behavior and uniqueness of the Jost solution gives

$$\frac{\partial\chi}{\partial t} + iB_u\chi = i\sqrt{3}(\frac{E}{2} - \zeta)^2\chi, \tag{4.1}$$

for χ determined by the Jost solution ψ_1 or φ_1, and

$$\frac{\partial\chi}{\partial t} + iB_u\chi = i\sqrt{3}(\frac{E}{2} + \zeta)^2\chi, \tag{4.2}$$

for χ determined by the Jost solution ψ_2 or φ_2.

Differentiating (3.2) with respect to t, we get by using (4.1) and (4.2) that

$$\frac{da}{dt}\psi_1 + (\frac{db}{dt} + i4\lambda\zeta b)\psi_2 = 0.$$

So, we get the evolution equations of the transition coefficients

$$a(t,\lambda) = a(0,\lambda), \qquad b(t,\lambda) = b(0,\lambda)\exp(-i4\lambda\zeta t). \tag{4.3}$$

Thus, the zero of $a(t,\lambda)$ does not depend on the time t.

At the zero λ_0 of $a(\lambda)$, we have

$$\varphi_1(x,\lambda_0) = b_0\psi_2(x,\lambda_0).$$

We get by using the evolution equations (4.1) and (4.2) of the Jost solutions that

$$\frac{db_0(t)}{dt} + i4\lambda_0\zeta_0 b_0(t) = 0,$$

which gives

$$b_0(t) = b_0(0)\exp(-i4\lambda_0\zeta_0 t), \quad \zeta_0 = (\lambda_0^2 - \frac{1}{2})^{\frac{1}{2}}. \tag{4.4}$$

5 The Marchenko equations

In this section, we will derive the Marchenko equations which connect the potential $q(x)$ with the scattering data. Throughout this section, we assume that the function $a(\lambda)$ has one zero λ_0, and we denote by b_0 the constant such that

$$\varphi_1(x,\lambda_0) = b_0\psi_2(x,\lambda_0), \tag{5.1}$$

and

$$\nu_0 = \sqrt{\frac{1}{2} - \lambda_0^2}\ ,\ \mu_0 = \frac{b_0}{\nu_0 a'(\lambda_0)}.$$

From (3.16), we know that μ_0 is a real number. Moreover, by the the uniqueness of the Jost solution, the change $(\lambda,\zeta) \in H \mapsto (\lambda,-\zeta) \in H$ implies that

$$\varphi_1 \mapsto \frac{1}{\sqrt{2}(\lambda-\zeta)}\varphi_2, \qquad \psi_1 \mapsto \frac{1}{\sqrt{2}(\lambda-\zeta)}\psi_2,$$

which together with (3.2) implies that

$$a \mapsto -a^*, \qquad b \mapsto -b^*. \tag{5.2}$$

For $\zeta \in \mathbb{R}$, we set

$$\lambda = \lambda(\zeta) = \sqrt{\zeta^2 + \frac{1}{2}}\ ,$$

and

$$c_1(\zeta) = c(\lambda) + c(-\lambda), \qquad c_2(\zeta) = \frac{c(\lambda) - c(-\lambda)}{\lambda}, \quad c = \frac{b}{a}.$$

The above observations imply that

$$c_1(-\zeta) = c_1(\zeta)^*\ ,\ c_2(-\zeta) = c_2(\zeta)^*\ .$$

We then define the following real valued functions of the real variable z,

$$F_1(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_1(\zeta)e^{i\zeta z}d\zeta - \mu_0\lambda_0 e^{-\nu_0 z}\ ,\ F_2(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_2(\zeta)e^{i\zeta z}d\zeta - \mu_0 e^{-\nu_0 z}\ . \tag{5.3}$$

Proposition 5.1. *The functions Ψ_{11}, Ψ_{12} defined by (2.11), (2.12) satisfy the following system for $y \geq x$,*

$$2\sqrt{2}\Psi_{11}(x,y) = F_2(x+y) - \int_x^{+\infty} \Psi_{12}(x,s)\sqrt{2}(F_1(s+y) - iF_2'(s+y)) + \Psi_{11}(x,s)F_2(s+y)ds \ , \tag{5.4}$$

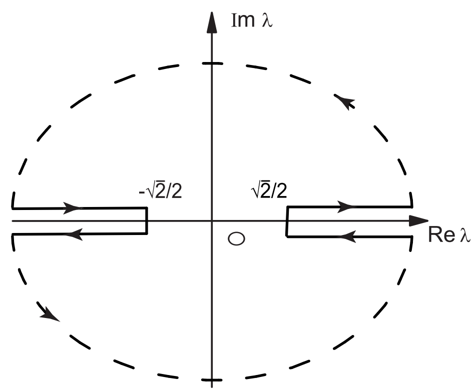
and

$$2\sqrt{2}\Psi_{12}(x,y) = \sqrt{2}(F_1(x+y) + iF_2'(x+y)) - \int_x^{+\infty} \Psi_{11}(x,s)\sqrt{2}(F_1(s+y) + iF_2'(s+y)) + \Psi_{12}(x,s)F_2(s+y)ds \ . \tag{5.5}$$

Let us prove Proposition 5.1. We rewrite the first equation of (3.2) in the form

$$\left(\frac{\varphi_1}{a} - X_1^+\right)\frac{e^{i\zeta y}}{2\pi\zeta} = \left(\psi_1 - X_1^+ + \frac{b}{a}\psi_2\right)\frac{e^{i\zeta y}}{2\pi\zeta}, \quad y \geq x. \tag{5.6}$$

The left-hand side of (5.6) is analytic on the upper sheet Γ^+ of the Riemann surface Γ, with the exception of the point λ_0. We integrate (5.6) along the contour with respect to λ indicated in the following figure:



We get by using the residue theorem and (5.6) that

$$\int \left(\frac{1}{a}\varphi_1 - X_1^+\right) e^{i\zeta y}\frac{d\lambda}{2\pi\zeta} = \frac{\varphi_1(x,\lambda_0)}{\nu_0 a'(\lambda_0)}e^{-\nu_0 y} = b_0\frac{\psi_2(x,\lambda_0)}{\nu_0 a'(\lambda_0)}e^{-\nu_0 y}. \tag{5.7}$$

Substituting the representation (2.14) into (5.7), we get

$$\int \left(\frac{1}{a}\varphi_1 - X_1^+\right) e^{i\zeta y}\frac{d\lambda}{2\pi\zeta} = \begin{pmatrix} \sqrt{2}F_1^{(1)}(x+y) + i\sqrt{2}F_2^{(1)'}(x+y) \\ F_2^{(1)}(x+y) \end{pmatrix} - \int_x^{\infty} \Psi(x,s)\begin{pmatrix} \sqrt{2}F_1^{(1)}(s+y) + i\sqrt{2}F_2^{(1)'}(s+y) \\ F_2^{(1)}(s+y) \end{pmatrix} ds \ , \tag{5.8}$$

where

$$F_1^{(1)}(z) = \mu_0\lambda_0 e^{-\nu_0 z}, \quad F_2^{(1)}(z) = \mu_0 e^{-\nu_0 z} \ . \tag{5.9}$$

On the other hand, using the representations (2.10) and (2.14), the contribution of the right hand side of (5.6) to the integral reads

$$\int \left(\psi_1 - X_1^+ + \frac{b}{a}\psi_2\right) e^{i\zeta y}\frac{d\lambda}{2\pi\zeta} = -\int_x^{+\infty} \Psi(x,s)\int X_1^+(s,\lambda)e^{i\zeta y}\frac{d\lambda}{2\pi\zeta}\,ds$$
$$+\int e^{i\zeta y}X_2^+(x,\lambda)\frac{b}{a}(\lambda)\frac{d\lambda}{2\pi\zeta} - \int_x^{+\infty}\Psi(x,s)\int e^{i\zeta y}X_2^+(s,\lambda)\frac{b}{a}(\lambda)\frac{d\lambda}{2\pi\zeta}\,ds.$$

Note that, if f is holomorphic and bounded on $\overline{\Gamma}^+$,

$$\int e^{i\zeta y}f(\lambda)\frac{d\lambda}{2\pi\zeta} = \int_{-\infty}^{+\infty} e^{i\zeta y}\frac{f(\lambda(\zeta))-f(-\lambda(\zeta))}{\lambda(\zeta)}\frac{d\zeta}{2\pi}\ . \tag{5.10}$$

Then, we get by using (5.10) that

$$\int e^{i\zeta y}\left(\psi_1 - X_1^+ + \frac{b}{a}\psi_2\right)\frac{d\lambda}{2\pi\zeta} = -\Psi(x,y)\begin{pmatrix}0\\ 2\sqrt{2}\end{pmatrix}$$
$$+\begin{pmatrix}\sqrt{2}F_1^{(2)}(x+y)+i\sqrt{2}F_2^{(2)'}(x+y)\\ F_2^{(2)}(x+y)\end{pmatrix}$$
$$-\int_x^{\infty}\Psi(x,s)\begin{pmatrix}\sqrt{2}F_1^{(2)}(s+y)+i\sqrt{2}F_2^{(2)'}(s+y)\\ F_2^{(2)}(s+y)\end{pmatrix}ds\ , \tag{5.11}$$

where

$$F_1^{(2)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_1(\zeta)e^{i\zeta z}d\zeta\ , F_2^{(2)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_2(\zeta)e^{i\zeta z}d\zeta\ . \tag{5.12}$$

Combining (5.8) and (5.11), and recalling the symmetry properties (2.13) of Ψ, this completes the proof of Proposition 5.1. □

Marchenko equations stated in this proposition are related to the asymptotic analysis of Jost solutions for $x\to+\infty$ and for this reason are called Marchenko equations from the right. Marchenko equations from the left can be derived similarly. We state the result without proof. Set

$$\widetilde{c}_1(\zeta) = \widetilde{c}(\lambda)+\widetilde{c}(-\lambda), \qquad \widetilde{c}_2(\zeta) = \frac{\widetilde{c}(\lambda)-\widetilde{c}(-\lambda)}{\lambda}, \quad \widetilde{c} = -\frac{b^*}{a},$$

and define the following real-valued functions of the real variable z,

$$\widetilde{F}_1^{(1)}(z) = \mu_0\lambda_0 e^{\nu_0 z}\ ,\ \widetilde{F}_2^{(1)}(z) = \mu_0 e^{\nu_0 z}\ , \tag{5.13}$$

$$\widetilde{F}_1^{2}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widetilde{c}_1(\zeta)e^{i\zeta z}d\zeta\ ,\ \widetilde{F}_2^{(2)}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widetilde{c}_2(\zeta)e^{i\zeta z}d\zeta\ , \tag{5.14}$$

$$\widetilde{F}_1 = \widetilde{F}_1^{(2)} - \widetilde{F}_1^{(1)}\ ,\ \widetilde{F}_2 = \widetilde{F}_2^{(2)} - \widetilde{F}_2^{(1)}\ . \tag{5.15}$$

Proposition 5.2. *The functions Φ_{11}, Φ_{12} defined by (2.16), (2.17) satisfy the following system of equations,*

$$2\sqrt{2}\Phi_{11}(x,y) = -\widetilde{F}_2(x+y)$$
$$-\int_{-\infty}^{x}\Phi_{12}(x,s)\sqrt{2}(\widetilde{F}_1(s+y)-i\widetilde{F}_2'(s+y))+\Phi_{11}(x,s)\widetilde{F}_2(s+y)ds\ , \tag{5.16}$$

and

$$
\begin{aligned}
2\sqrt{2}\Phi_{12}(x,y) &= \sqrt{2}(\widetilde{F}_1(x+y) + i\widetilde{F}_2'(x+y)) \\
&- \int_{-\infty}^{x} \Phi_{11}(x,s)\sqrt{2}(\widetilde{F}_1(s+y) + i\widetilde{F}_2'(s+y)) + \Phi_{12}(x,s)\widetilde{F}_2(s+y)ds \ , \qquad (5.17)
\end{aligned}
$$

for $y \le x$.

In particular, in the unperturbed case case when $q(x) = \frac{\sqrt{2}}{2}\tanh(\frac{x}{\sqrt{2}})$, we have

$$
a(\lambda) = \frac{\lambda + \zeta - \frac{\sqrt{2}}{2}i}{\lambda + \zeta + \frac{\sqrt{2}}{2}i}, \quad b(\lambda) = 0, \quad \lambda_0 = 0, \quad \mu_0 = -2. \qquad (5.18)
$$

Thus, $F_1(z) = 0,\ F_2(z) = 2e^{-\frac{\sqrt{2}}{2}z}$. The Marchenko equations become

$$
\begin{aligned}
2\sqrt{2}\Psi_{11}(x,y) &= 2e^{-\frac{\sqrt{2}}{2}(x+y)} \\
&-2i\int_x^{+\infty} \Psi_{12}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds - 2\int_x^{+\infty} \Psi_{11}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds, \qquad (5.19)
\end{aligned}
$$

and

$$
\begin{aligned}
2\sqrt{2}\Psi_{12}(x,y) &= -2ie^{-\frac{\sqrt{2}}{2}(x+y)} \\
&+2i\int_x^{+\infty} \Psi_{11}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds - 2\int_x^{+\infty} \Psi_{12}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds \ , \qquad (5.20)
\end{aligned}
$$

which allows to recover the values of $\Psi^0(x,y)$ already found in subsection 2.3. (see (2.24)).

6 Further properties of the transition coefficients

In this section, we will use the Marchenko equations to obtain the further information about the transition coefficients which will play an important role in the proof of Theorem 1.1. In section 3, we proved that $a(\lambda)$ has at most one zero. In the following, we will prove the existence of zero.

Proposition 6.1. *The function $a(\lambda)$ has exactly one zero.*

Proof. By Lemma 3.2, we know that $a(\lambda)$ has at most one zero λ_0, and that $|\lambda_0| \le C\varepsilon$. It remains to prove the existence of this zero. Assume that $a(\lambda)$ does not vanish. In this case, the Marchenko equations become

$$
\begin{aligned}
2\sqrt{2}\Psi_{11}(x,y) &= F_2^{(2)}(x+y) \\
&- \int_x^{+\infty} \Psi_{12}(x,s)\sqrt{2}(F_1^{(2)}(s+y) - iF_2^{(2)'}(s+y)) + \Psi_{11}(x,s)F_2^{(2)}(s+y)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
2\sqrt{2}\Psi_{12}(x,y) &= \sqrt{2}(F_1^{(2)}(x+y) + iF_2^{(2)'}(x+y)) \\
&- \int_x^{+\infty} \Psi_{11}(x,s)\sqrt{2}(F_1^{(2)}(s+y) + iF_2^{(2)'}(s+y)) + \Psi_{12}(x,s)F_2^{(2)}(s+y)ds.
\end{aligned}
$$

Thus, we get by the Young inequality that

$$\|\Psi(0,\cdot)\|_{L^2(y\geq 0)} \leq C(1+\|\Psi(0,.)\|_{L^1(y\geq 0)})(\|F_1^{(2)}\|_{L^2}+\|F_2^{(2)}\|_{L^2}+\|F_2^{(2)'}\|_{L^2}). \quad (6.1)$$

We get by (2.21) and Lemma 2.4 that

$$\|\Psi(0,.)\|_{L^1(y\geq 0)} \leq C, \quad \|\Psi(0,\cdot)-\Psi^0(0,\cdot)\|_{L^2(y\geq 0)} \leq C\varepsilon \ , \quad (6.2)$$

where Ψ^0 denotes the unperturbed kernel, given by (2.24). By Lemma 3.1, we have

$$\|F_1^{(2)}\|_{L^2}+\|F_2^{(2)}\|_{L^2}+\|F_2^{(2)'}\|_{L^2} \leq C\varepsilon^{\frac{1}{2}}. \quad (6.3)$$

Indeed, by the Plancherel formula,

$$\|F_1^{(2)}\|_{L^2}^2 \leq C\int_{\mathbb{R}}\left|\frac{b(\lambda)}{a(\lambda)}\right|^2 d\zeta = C\int_{|\zeta|\leq\varepsilon} d\zeta + C\varepsilon^2\int_{|\zeta|\geq\varepsilon}\frac{1}{\zeta^2}d\zeta \leq C\varepsilon.$$

By summing up (6.1)-(6.3), we obtain

$$\frac{\sqrt{2}}{4} = \|\Psi^0(0,\cdot)\|_{L^2(y\geq 0)}^2 \leq C\varepsilon,$$

which is impossible. So, $a(\lambda)$ must have a zero. □

Proposition 6.2. *If ε is small enough, then the transition coefficients $a(\lambda), b(\lambda)$ satisfy, on H,*

$$\lim_{\zeta\to 0}\zeta a(\lambda)=0, \quad \lim_{\zeta\to 0}\zeta b(\lambda)=0. \quad (6.4)$$

In order to prove Proposition 6.2, we need the following two lemmas.

Lemma 6.1. *Let $\{F_1^{(1)}(x), F_2^{(1)}(x)\}$ be given by (5.9), and $\{\widetilde{F}_1^{(1)}(x), \widetilde{F}_2^{(1)}(x)\}$ be given by (5.13). Then there hold*

$$|F_1^{(1)}(x)|+|F_2^{(1)}(x)+2e^{-\frac{\sqrt{2}}{2}x}|+|F_2^{(1)'}(x)-\sqrt{2}e^{-\frac{\sqrt{2}}{2}x}| \leq C\varepsilon e^{-\frac{1}{2}x} \quad \textit{for } x\geq 0,$$
$$|\widetilde{F}_1^{(1)}(x)|+|\widetilde{F}_2^{(1)}(x)+2e^{\frac{\sqrt{2}}{2}x}|+|\widetilde{F}_2^{(1)'}(x)+\sqrt{2}e^{\frac{\sqrt{2}}{2}x}| \leq C\varepsilon e^{\frac{1}{2}x} \quad \textit{for } x\leq 0.$$

Proof. Lemma 6.1 is a simple consequence of Lemma 3.2. □

Lemma 6.2. *Let $\{F_1^{(2)}(x), F_2^{(2)}(x)\}$ be given by (5.12), and $\{\widetilde{F}_1^{(2)}(x), \widetilde{F}_2^{(2)}(x)\}$ be given by (5.14). Then there exists $M>0$ (independent of ε) such that*

$$\int_M^{+\infty}\left(|F_1^{(2)}(x)|+|F_2^{(2)}(x)|+|F_2^{(2)'}(x)|\right)dx \leq C\varepsilon \quad , \quad (6.5)$$

$$\int_{-\infty}^{-M}\left(|\widetilde{F}_1^{(2)}(x)|+|\widetilde{F}_2^{(2)}(x)|+|\widetilde{F}_2^{(2)'}(x)|\right)dx \leq C\varepsilon \ . \quad (6.6)$$

Proof. We just prove (6.5), since (6.6) can be similarly proved. We use the notation of Lemma 2.4. Substracting the Marchenko equations for Ψ^0 from the Marchenko equations for Ψ, we obtain

$$\begin{aligned}
2\varepsilon\sqrt{2}\Psi^1_{11}(x,y) &= F_2^{(2)}(x+y) - (F_2^{(1)}(x+y) + 2e^{-\frac{\sqrt{2}}{2}(x+y)})\\
&- \int_x^{+\infty} \Psi_{12}(x,s)\sqrt{2}(F_1^{(2)}(s+y) - iF_2^{(2)'}(s+y)) + \Psi_{11}(x,s)F_2^{(2)}(s+y)ds\\
&+ \int_x^{+\infty} \Psi_{12}(x,s)[F_1^{(1)}(s+y) + i(2e^{-\frac{\sqrt{2}}{2}(y+s)} - \sqrt{2}F_2^{(1)'}(s+y))]ds\\
&+ \int_x^{+\infty} \Psi_{11}(x,s)(2e^{-\frac{\sqrt{2}}{2}(y+s)} + F_2^{(1)}(s+y))ds\\
&- \int_x^{+\infty} 2i\varepsilon\Psi^1_{12}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)} + 2\varepsilon\Psi^1_{11}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds,
\end{aligned}$$

and

$$\begin{aligned}
&2\varepsilon\sqrt{2}\Psi^1_{12}(x,y) = \sqrt{2}(F_1^{(2)} + iF_2^{(2)'})(x+y) - \sqrt{2}[F_1^{(1)}(x+y) + i(F_2^{(1)'}(x+y) - \sqrt{2}e^{-\frac{\sqrt{2}}{2}(x+y)})]\\
&- \int_x^{+\infty} \Psi_{11}(x,s)\sqrt{2}(F_1^{(2)}(s+y) + iF_2^{(2)'}(s+y)) + \Psi_{12}(x,s)F_2^{(2)}(s+y)ds\\
&+ \int_x^{+\infty} \Psi_{11}(x,s)[F_1^{(1)} - i(2e^{-\frac{\sqrt{2}}{2}(y+s)} - \sqrt{2}F_2^{(1)'}(s+y))]ds\\
&+ \int_x^{+\infty} \Psi_{12}(x,s)(2e^{-\frac{\sqrt{2}}{2}(y+s)} + F_2^{(1)}(s+y))ds\\
&+ \int_x^{+\infty} 2i\varepsilon\Psi^1_{11}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)} - 2\varepsilon\Psi^1_{12}(x,s)e^{-\frac{\sqrt{2}}{2}(y+s)}ds.
\end{aligned}$$

Integrating in $y \in [x, +\infty[$, and using Lemma 2.4 and Lemma 6.1, we infer, for every $x \geq 0$,

$$\begin{aligned}
&\int_{2x}^{+\infty} \left(|F_1^{(2)}(z)| + |F_2^{(2)}(z)| + |F_2^{(2)'}(z)|\right) dz \leq C\varepsilon\\
&\quad + C\int_0^{+\infty} |\underline{\Psi}(x,p)| \int_{2x+2p}^{+\infty} \left(|F_1^{(2)}(z)| + |F_2^{(2)}(z)| + |F_2^{(2)'}(z)|\right) dz\, dp.
\end{aligned}$$

In view of estimate (2.21), this completes the proof of Lemma 6.2 $\square$

Now we are in position to complete the proof of Proposition 6.2. We can expand $a(\lambda)$ and $b(\lambda)$ near $\zeta = 0$ as

$$a(\lambda) = a^0(\zeta) + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta), \quad b(\lambda) = \frac{\sigma_+(\varepsilon)}{\zeta} + b_1(\zeta) \quad \text{for } \lambda > 0, \tag{6.7}$$

$$a(\lambda) = a^0(\zeta) + \frac{\sigma_-(\varepsilon)}{\zeta} + a_1(\zeta), \quad b(\lambda) = -\frac{\sigma_-(\varepsilon)}{\zeta} + b_1(\zeta) \quad \text{for } \lambda < 0. \tag{6.8}$$

Here $a^0(\zeta) = \frac{\lambda+\zeta-\frac{\sqrt{2}}{2}i}{\lambda+\zeta+\frac{\sqrt{2}}{2}i}$, and from Lemma 2.5 and (5.2), we know that $\sigma_\pm(\varepsilon)$ is a real constant and analytic in ε (tending to zero as $\varepsilon \to 0$), $a_1(\zeta), b_1(\zeta) \in C^1(\mathbb{R})$ and for $k \leq 1$

$$|(\frac{d}{d\zeta})^k a_1(\zeta)| + |(\frac{d}{d\zeta})^k b_1(\zeta)| \leq C\varepsilon. \tag{6.9}$$

From (6.7) and (6.8), we find that if $\sigma_\pm(\varepsilon) \neq 0$, we have

$$\lim_{\lambda \to \frac{\sqrt{2}}{2}} \frac{b(\lambda)}{a(\lambda)} = 1, \qquad \lim_{\lambda \to -\frac{\sqrt{2}}{2}} \frac{b(\lambda)}{a(\lambda)} = -1. \tag{6.10}$$

Lemma 6.3. *Let $\lambda = (\frac{1}{2} + \zeta^2)^{\frac{1}{2}}$. There exists a constant C independent of δ_0 and ε such that as ε tends to zero,*

$$|\lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{c(\lambda)}{\zeta} d\zeta - i sgn\, \sigma_+(\varepsilon)| \leq C\delta_0, \tag{6.11}$$

$$|\lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{c(-\lambda)}{\zeta} d\zeta + i sgn\, \sigma_-(\varepsilon)| \leq C\delta_0, \tag{6.12}$$

for $\delta_0 > 0$ small enough.

Proof. We first prove (6.11). We get by making an expansion for $a^0(\zeta)$ near $\zeta = 0$ that

$$a(\lambda) = -i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta),$$

where $a_2(\zeta)$ is smooth and $a_2(0) = 0$. Thus, we get by (6.7) that

$$\begin{aligned}
\lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{c(\lambda)}{\zeta} d\zeta &= \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{\frac{\sigma_+}{\zeta} + b_1(\zeta)}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)}}{\zeta} d\zeta \\
&= \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{\frac{\sigma_+}{\zeta} + b_1(\zeta)}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)} - 1}{\zeta} d\zeta \\
&= \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{i + b_1(\zeta) - a_1(\zeta) - a_2(\zeta)}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)}}{\zeta} d\zeta.
\end{aligned} \tag{6.13}$$

Obviously, if $f(\zeta) = 0(|\zeta|)$, then there holds for δ_0 small enough

$$|\int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{f(\zeta)}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)}}{\zeta} d\zeta| \leq C\delta_0,$$

which together with (6.9) implies that

$$|\lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{c(\lambda)}{\zeta} d\zeta - \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{i + b_1(0) - a_1(0)}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)}}{\zeta} d\zeta| \leq C\delta_0. \tag{6.14}$$

Now, we write

$$\begin{aligned}
&\lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{1}{-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta)}}{\zeta} d\zeta \\
&= \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{-a_1(\zeta) - a_2(\zeta)}{(-i + \frac{\sigma_+(\varepsilon)}{\zeta} + a_1(\zeta) + a_2(\zeta))(-i + \frac{\sigma_+(\varepsilon)}{\zeta})}}{\zeta} d\zeta + \lim_{\delta \to 0} \frac{1}{\pi} \int_{\delta \leq |\zeta| \leq \delta_0} \frac{\frac{1}{-i + \frac{\sigma_+(\varepsilon)}{\zeta}}}{\zeta} d\zeta
\end{aligned} \tag{6.15}$$

By a direct computation, we get for $\sigma_+(\varepsilon)\neq 0$

$$\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{\frac{1}{-i+\frac{\sigma_+(\varepsilon)}{\zeta}}}{\zeta}d\zeta=\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{i\zeta+\sigma_+(\varepsilon)}{\zeta^2+\sigma_+(\varepsilon)^2}d\zeta$$
$$=\lim_{\delta\to 0}\frac{2}{\pi}\int_{\delta}^{\delta_0}\frac{\sigma_+(\varepsilon)}{\zeta^2+\sigma_+(\varepsilon)^2}d\zeta$$
$$=\frac{2}{\pi}\int_0^{\frac{\delta_0}{\sigma_+(\varepsilon)}}\frac{1}{1+\zeta^2}d\zeta=\frac{2}{\pi}\arctan(\frac{\delta_0}{\sigma_+(\varepsilon)}),\tag{6.16}$$

and for $\sigma_+(\varepsilon)=0$

$$\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{\frac{1}{-i+\frac{\sigma_+(\varepsilon)}{\zeta}}}{\zeta}d\zeta=0.\tag{6.17}$$

By the properties of $a_1(\zeta)$ and $a_2(\zeta)$, it is easy to prove that for $\sigma_+(\varepsilon)\neq 0$

$$|\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{\frac{-a_1(\zeta)-a_2(\zeta)}{(-i+\frac{\sigma_+(\varepsilon)}{\zeta}+a_1(\zeta)+a_2(\zeta))(-i+\frac{\sigma_+(\varepsilon)}{\zeta})}}{\zeta}d\zeta|\le C\delta_0+C\varepsilon\ln\sigma_+(\varepsilon),\tag{6.18}$$

and for $\sigma_+(\varepsilon)=0$

$$|\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{\frac{-a_1(\zeta)-a_2(\zeta)}{(-i+\frac{\sigma_+(\varepsilon)}{\zeta}+a_1(\zeta)+a_2(\zeta))(-i+\frac{\sigma_+(\varepsilon)}{\zeta})}}{\zeta}d\zeta|\le C\delta_0.\tag{6.19}$$

Note that if $\sigma_+(\varepsilon)\neq 0$, we have, by the analyticity of $\sigma_+(\varepsilon)$,

$$\lim_{\varepsilon\to 0}\varepsilon\ln\sigma_+(\varepsilon)=0.$$

By summing up (6.13)-(6.19), we get that as ε tends to zero

$$|\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\le|\zeta|\le\delta_0}\frac{c(\lambda)}{\zeta}d\zeta-i\mathrm{sgn}\,\sigma_+(\varepsilon)|\le C\delta_0,$$

for δ_0 small enough. Since (6.12) can be similarly proved, we omit its proof here. □

To complete the proof of Proposition 6.2, it is sufficient to prove that $\sigma_\pm(\varepsilon)$ must be zero.

Case 1. $\sigma_+(\varepsilon)>0$.

By the definition of $F_2^{(2)}(x)$, we have

$$\int_M^{+\infty}F_2^{(2)}(x)dx=\frac{1}{2}(\check{c}_2(0)-i\mathrm{H}\check{c}_2(0)),\tag{6.20}$$

where H is the Hilbert transform and

$$\check{c}_2(\zeta)=e^{iM\zeta}\frac{c(\lambda)-c(-\lambda)}{\lambda},\quad c=\frac{b}{a}.$$

We get by Lemma 6.2 that

$$|\breve{c}_2(0) - i\mathrm{H}\breve{c}_2(0)| \leq C\varepsilon. \tag{6.21}$$

By the definition of the Hilbert transform, we have

$$\mathrm{H}\breve{c}_2(0) = \lim_{\delta\to 0}\frac{1}{\pi}\int_{|\zeta|\geq\delta}\frac{\breve{c}_2(\zeta)}{\zeta}d\zeta = \lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}\frac{\breve{c}_2(\zeta)}{\zeta}d\zeta + \frac{1}{\pi}\int_{|\zeta|\geq\delta_0}\frac{\breve{c}_2(\zeta)}{\zeta}d\zeta, \tag{6.22}$$

where δ_0 is a small enough constant. We get by (3.8) that

$$|\frac{1}{\pi}\int_{|\zeta|\geq\delta_0}\frac{\breve{c}_2(\zeta)}{\zeta}d\zeta| \leq C\varepsilon/\delta_0. \tag{6.23}$$

We write

$$\begin{aligned}\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}\frac{\breve{c}_2(\zeta)}{\zeta}d\zeta = &\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}(\frac{e^{iM\zeta}}{\lambda}-\sqrt{2})\frac{c(\lambda)-c(-\lambda)}{\zeta}d\zeta\\ &+\sqrt{2}\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}\frac{c(\lambda)-c(-\lambda)}{\zeta}d\zeta.\end{aligned}$$

Since $|c(\lambda)| \leq 1$ by (3.8), we get

$$|\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}(\frac{e^{iM\zeta}}{\lambda}-\sqrt{2})\frac{c(\lambda)-c(-\lambda)}{\zeta}d\zeta| \leq C\delta_0. \tag{6.24}$$

From Lemma 6.3, we find that as $\varepsilon \to 0$

$$|\lim_{\delta\to 0}\frac{1}{\pi}\int_{\delta\leq|\zeta|\leq\delta_0}\frac{c(\lambda)-c(-\lambda)}{\zeta}d\zeta - i(1+\operatorname{sgn}\sigma_-(\varepsilon))| \leq C\delta_0. \tag{6.25}$$

By summing up (6.22)-(6.25), we get that as $\varepsilon \to 0$

$$|\mathrm{H}\breve{c}_2(0) - i\sqrt{2}(1+\operatorname{sgn}\sigma_-(\varepsilon))| = o(1). \tag{6.26}$$

On the other hand, we get by (6.8) and (6.10) that as $\varepsilon \to 0$

$$|\breve{c}_2(0) - \sqrt{2}(1+|\operatorname{sgn}\sigma_-(\varepsilon)|)| = o(1),$$

from which and (6.26), it follows that, as $\varepsilon \to 0$,

$$|\breve{c}_2(0) - i\mathrm{H}\breve{c}_2(0)| = |2\sqrt{2}+\sqrt{2}(|\operatorname{sgn}\sigma_-(\varepsilon)|+\operatorname{sgn}\sigma_-(\varepsilon))| + o(1) \geq \sqrt{2},$$

which contradicts (6.21).

Case 2. $\sigma_-(\varepsilon) > 0$.

Exactly as in the proof of Case 1, we can get, as $\varepsilon \to 0$,

$$|\breve{c}_2(0) - i\mathrm{H}\breve{c}_2(0)| = |2\sqrt{2}+\sqrt{2}(|\operatorname{sgn}\sigma_+(\varepsilon)|+\operatorname{sgn}\sigma_+(\varepsilon))| + o(1) \geq \sqrt{2},$$

which contradicts (6.21).

Case 3. $\sigma_+(\varepsilon) < 0$.

By the definition of $\widetilde{F}_2^{(2)}(x)$, we have

$$\int_{-\infty}^{-M} \widetilde{F}_2^{(2)}(x)dx = \frac{1}{2}(\breve{\widetilde{c}}_2(0) + i\mathrm{H}\breve{\widetilde{c}}_2(0)),$$

where H is the Hilbert transform and

$$\breve{\widetilde{c}}_2(\zeta) = e^{-iM\zeta}\frac{\widetilde{c}(\lambda) - \widetilde{c}(-\lambda)}{\lambda}, \quad \widetilde{c} = -\frac{b^*}{a}.$$

We get by (6.6) that

$$|\breve{\widetilde{c}}_2(0) + i\mathrm{H}\breve{\widetilde{c}}_2(0)| \leq C\varepsilon. \tag{6.27}$$

Exactly as in the proof of Case 1, we can get, as $\varepsilon \to 0$,

$$\begin{aligned} &|\breve{\widetilde{c}}_2(0) + \sqrt{2}(1 + |\mathrm{sgn}\,\sigma_-(\varepsilon)|)| = o(1), \\ &|\mathrm{H}\breve{\widetilde{c}}_2(0) - i\sqrt{2}(1 - \mathrm{sgn}\,\sigma_-(\varepsilon))| = o(1). \end{aligned}$$

So, we have

$$|\breve{\widetilde{c}}_2(0) + i\mathrm{H}\breve{\widetilde{c}}_2(0)| = |2\sqrt{2} + \sqrt{2}(|\mathrm{sgn}\,\sigma_-(\varepsilon)| - \mathrm{sgn}\,\sigma_-(\varepsilon))| + o(1) \geq \sqrt{2},$$

which contradicts (6.27).

Case 4. $\sigma_-(\varepsilon) < 0$.

As in case 3, we have, as $\varepsilon \to 0$,

$$|\breve{\widetilde{c}}_2(0) + i\mathrm{H}\breve{\widetilde{c}}_2(0)| = |2\sqrt{2} + \sqrt{2}(|\mathrm{sgn}\,\sigma_+(\varepsilon)| - \mathrm{sgn}\,\sigma_+(\varepsilon))| + o(1) \geq \sqrt{2},$$

which contradicts (6.27).

So, we conclude that $\sigma_\pm$ must be zero. This completes the proof of Proposition 6.2. □

As a corollary of Proposition 6.2, we have

Proposition 6.3. *For ε small enough, we have*

$$|\frac{b(\lambda)}{a(\lambda)}| \leq C\varepsilon\langle\zeta\rangle^{-3}. \tag{6.28}$$

Proof. By Lemma 3.1, it suffices to prove that

$$|\frac{b(\lambda)}{a(\lambda)}| \leq C\varepsilon,$$

which can be deduced from (6.7) and (6.8), since $\sigma_\pm(\varepsilon)$ is zero by Proposition 6.2. □

7 Proof of Theorem 1.1

Let $\{a(\lambda), b(\lambda)\}$ be given by (3.3). By Proposition 6.1, $a(\lambda)$ has only one zero λ_0. Let b_0 be the constant such that

$$\varphi_1(x, \lambda_0) = b_0 \psi_2(x, \lambda_0).$$

The evolution transition coefficients $\{a(t,\lambda), b(t,\lambda)\}$ are given by

$$a(t, \lambda) = a(\lambda), \quad b(t, \lambda) = b(\lambda) \exp(-i4\lambda\zeta t).$$

and

$$\mu_0(t) = \frac{b_0(t)}{\nu_0 a'(\lambda_0)}, \quad b_0(t) = b_0 \exp(4\lambda_0\nu_0 t), \quad \zeta_0 = (\lambda_0^2 - \frac{1}{2})^{\frac{1}{2}} = i\nu_0.$$

We set

$$\begin{aligned}
&F_1^{(1)}(t,z) = \mu_0(t)\lambda_0 e^{-\nu_0 z}, \quad F_2^{(1)}(t,z) = \mu_0(t) e^{-\nu_0 z}, \\
&F_1^{(2)}(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_1(t,\zeta) e^{i\zeta z} d\zeta, \quad F_2^{(2)}(t,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_2(t,\zeta) e^{i\zeta z} d\zeta, \\
&c_1(t,\zeta) = c(t,\lambda) + c(t,-\lambda), \qquad c_2(t,\zeta) = \frac{c(t,\lambda) - c(t,-\lambda)}{\lambda}, \quad c(t,\lambda) = \frac{b(t,\lambda)}{a(t,\lambda)}, \\
&F_1(t,z) = F_1^{(2)}(t,z) - F_1^{(1)}(t,z), \qquad F_2(z) = F_2^{(2)}(t,z) - F_2^{(1)}(t,z).
\end{aligned}$$

In the following, we will solve the evolution Marchenko equations for $y \geq x$

$$\begin{aligned}
&2\sqrt{2}\Psi_{11}(t,x,y) = F_2(t,x+y) \\
&\quad - \int_x^{+\infty} \Psi_{12}(t,x,s)\sqrt{2}(F_1(t,s+y) - iF_2'(t,s+y)) + \Psi_{11}(t,x,s)F_2(t,s+y)ds, \quad (7.1)
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{2}\Psi_{12}(t,x,y) = \sqrt{2}(F_1(t,x+y) + iF_2'(t,x+y)) \\
&\quad - \int_x^{+\infty} \Psi_{11}(t,x,s)\sqrt{2}(F_1(t,s+y) + iF_2'(t,s+y)) + \Psi_{12}(t,x,s)F_2(t,s+y)ds. \quad (7.2)
\end{aligned}$$

Let $\{\Psi_{11}^{(1)}(t,x,y), \Psi_{12}^{(1)}(t,x,y)\}$ be the solution of the following Marchenko equations involving only kernel $F^{(1)}$, namely

$$\begin{aligned}
&2\sqrt{2}\Psi_{11}^{(1)}(t,x,y) = -F_2^{(1)}(t,x+y) \\
&\quad + \int_x^{+\infty} \Psi_{12}^{(1)}(t,x,s)\sqrt{2}(F_1^{(1)}(t,s+y) - iF_2^{(1)'}(t,s+y)) + \Psi_{11}^{(1)}(t,x,s)F_2^{(1)}(t,s+y)ds,
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{2}\Psi_{12}^{(1)}(t,x,y) = -\sqrt{2}(F_1^{(1)}(t,x+y) + iF_2^{(1)'}(t,x+y)) \\
&\quad + \int_x^{+\infty} \Psi_{11}^{(1)}(t,x,s)\sqrt{2}(F_1^{(1)}(t,s+y) + iF_2^{(1)'}(t,s+y)) + \Psi_{12}^{(1)}(t,x,s)F_2^{(1)}(t,s+y)ds.
\end{aligned}$$

Notice that, in view of the expression of $F_1^{(1)}, F_2^{(1)}$ recalled above, these integral are finite rank equations. Consequently, they can be exactly solved as

$$\Psi_{11}^{(1)}(t,x,y) = \frac{\nu_0 e^{\nu_0(x-y)}}{1 - \frac{2\sqrt{2}\nu_0}{\mu_0(t)} e^{2\nu_0 x}}, \quad \Psi_{12}^{(1)}(t,x,y) = \frac{\sqrt{2}\nu_0(\lambda_0 - i\nu_0) e^{\nu_0(x-y)}}{1 - \frac{2\sqrt{2}\nu_0}{\mu_0(t)} e^{2\nu_0 x}}. \tag{7.3}$$

Proposition 7.1. *There exists a unique solution* $\{\Psi_{11}(t,x,y), \Psi_{12}(t,x,y)\}$ *to the Marchenko equations (7.1) and (7.2) such that for* $|\alpha| \le 2$

$$\partial^\alpha \Psi_{11}(t,x,y), \quad \partial^\alpha \Psi_{12}(t,x,y) \in L^2_y(y \ge x), \tag{7.4}$$

$$\|\partial_y^\alpha \big(\Psi(t,x,\cdot) - \Psi^{(1)}(t,x,\cdot)\big)\|_{L^2(y\ge x)} \le C\varepsilon. \tag{7.5}$$

Furthermore, if we set

$$u(t,x) = 2\sqrt{2} i \Psi_{21}(t,x,x) + 1, \tag{7.6}$$

then $u(t,x)$ *is a solution of the Gross-Pitaevskii equation (1.1).*

Proof. We set

$$\Psi_{11}^r(t,x,y) = \Psi_{11}(t,x,y) - \Psi_{11}^{(1)}(t,x,y), \quad \Psi_{12}^r(t,x,y) = \Psi_{12}(t,x,y) - \Psi_{12}^{(1)}(t,x,y).$$

Then the Marchenko equations (7.1) and (7.2) are transformed into the equations in terms of $\{\Psi_{11}^r(t,x,y), \Psi_{12}^r(t,x,y)\}$

$$\begin{aligned} 2\sqrt{2}\Psi_{11}^r(t,x,y) = &\int_x^{+\infty} \Psi_{12}^r(t,x,s)\sqrt{2}(F_1^{(1)}(t,s+y) - iF_2^{(1)'}(t,s+y)) \\ &+ \Psi_{11}^r(t,x,s) F_2^{(1)}(t,s+y) ds + \mathcal{F}_1(t,x,y), \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} 2\sqrt{2}\Psi_{12}^r(t,x,y) = &\int_x^{+\infty} \Psi_{11}^r(t,x,s)\sqrt{2}(F_1^{(1)}(t,s+y) + iF_2^{(1)'}(t,s+y)) \\ &+ \Psi_{12}^r(t,x,s) F_2^{(1)}(t,s+y) ds + \mathcal{F}_2(t,x,y). \end{aligned} \tag{7.8}$$

Here

$$\begin{aligned} \mathcal{F}_1(t,x,y) = &-\int_x^{+\infty} (\Psi_{12}^{(1)} + \Psi_{12}^r)(t,x,s)\sqrt{2}(F_1^{(2)}(t,s+y) - iF_2^{(2)'}(t,s+y)) \\ &+ (\Psi_{11}^{(1)} + \Psi_{11}^r)(t,x,s) F_2^{(2)}(t,s+y) ds + F_2^{(2)}(t,x+y), \end{aligned} \tag{7.9}$$

$$\begin{aligned} \mathcal{F}_2(t,x,y) = &-\int_x^{+\infty} (\Psi_{11}^{(1)} + \Psi_{11}^r)(t,x,s)\sqrt{2}(F_1^{(2)}(t,s+y) + iF_2^{(2)'}(t,s+y)) \\ &+ (\Psi_{12}^{(1)} + \Psi_{12}^r)(t,x,s) F_2^{(2)}(t,s+y) ds + \sqrt{2}(F_1^{(2)}(t,x+y) + iF_2^{(2)'}(t,x+y)). \end{aligned} \tag{7.10}$$

For fixed $t, x \in \mathbb{R}$, if the source terms $\{\mathcal{F}_1(t,x,y), \mathcal{F}_1(t,x,y)\} \in L^2_y(y \ge x)$ are given, we firstly show that the integral equations (7.7) and (7.8) have a unique solution $\{\Psi_{11}^r(t,x,y), \Psi_{12}^r(t,x,y)\} \in L^2_y(y \ge x)$. We reformulate the integral equations (7.7) and (7.8) as

$$(I + \Omega_x)\Psi^r(t,x,y) = \mathcal{F}(t,x,y), \tag{7.11}$$

where

$$\Omega_x\Psi^r(t,x,y) = \int_x^{+\infty} \Omega(t,s+y)\Psi^r(t,x,s)ds.$$

As we already noticed, the kernel $\Omega(t,s)$ is finite rank. Therefore it suffices to show that the homogeneous equation

$$(I+\Omega_x)\Psi^r(t,x,y) = 0$$

has only a trivial solution in $L^2_y(y\geq x)$. Indeed, multiplying by $\overline{\Psi}^r_{11}(t,x,y)$ on both sides of (7.7), then integrating the resulting equations with respect to y on $(x,+\infty)$, we obtain

$$\begin{aligned}\int_x^{+\infty}|\Psi^r_{11}(t,x,y)|^2dy &= \mu_0(t)|\int_x^{+\infty}e^{-\nu_0 y}\Psi^r_{11}(t,x,y)dy|^2 + \int_x^{+\infty}\mathcal{F}_1(t,x,y)\overline{\Psi}^r_{11}(t,x,y)dy\\ &+\sqrt{2}\mu_0(t)(\lambda_0+i\nu_0)\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{12}(t,x,s)ds\int_x^{+\infty}e^{-\nu_0 y}\overline{\Psi}^r_{11}(t,x,y)dy. \qquad (7.12)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_x^{+\infty}|\Psi^r_{12}(t,x,y)|^2dy &= \mu_0(t)|\int_x^{+\infty}e^{-\nu_0 y}\Psi^r_{12}(t,x,y)dy|^2 + \int_x^{+\infty}\mathcal{F}_2(t,x,y)\overline{\Psi}^r_{12}(t,x,y)dy\\ &+\sqrt{2}\mu_0(t)(\lambda_0-i\nu_0)\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{11}(t,x,s)ds\int_x^{+\infty}e^{-\nu_0 y}\overline{\Psi}^r_{12}(t,x,y)dy. \qquad (7.13)\end{aligned}$$

We get by summing up (7.12) and (7.13) that

$$\begin{aligned}&2\sqrt{2}\int_x^{+\infty}(|\Psi^r_{11}(t,x,y)|^2+|\Psi^r_{12}(t,x,y)|^2)dy\\ &= \int_x^{+\infty}\mathcal{F}_1(t,x,y)\overline{\Psi}^r_{11}(t,x,y)dy + \int_x^{+\infty}\mathcal{F}_2(t,x,y)\overline{\Psi}^r_{12}(t,x,y)dy\\ &\quad+\mu_0(t)|\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{11}(t,x,y)dy|^2+\mu_0(t)|\int_x^{+\infty}e^{-\nu_0 y}\Psi^r_{12}(t,x,y)dy|^2\\ &\quad+2\sqrt{2}\mu_0(t)\lambda_0\mathcal{R}\Big(\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{12}(t,x,s)ds\int_x^{+\infty}e^{-\nu_0 y}\overline{\Psi}^r_{11}(t,x,y)dy\Big)\\ &\quad-2\sqrt{2}\mu_0(t)\nu_0\mathcal{I}\Big(\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{12}(t,x,s)ds\int_x^{+\infty}e^{-\nu_0 y}\overline{\Psi}^r_{11}(t,x,y)dy\Big). \qquad (7.14)\end{aligned}$$

Note that the last two terms on the right hand side of (7.14) are less than

$$\begin{aligned}&2\sqrt{2}\mu_0(t)(\lambda_0^2+\nu_0^2)^{\frac{1}{2}}\Big|\int_x^{+\infty}e^{-\nu_0 s}\Psi^r_{12}(t,x,s)ds\int_x^{+\infty}e^{-\nu_0 y}\overline{\Psi}^r_{11}(t,x,y)dy\Big|\\ &\leq \mu_0(t)\Big(|\int_x^{+\infty}e^{-\nu_0 y}\Psi^r_{11}(t,x,y)dy|^2+|\int_x^{+\infty}e^{-\nu_0 y}\Psi^r_{12}(t,x,y)dy|^2\Big).\end{aligned}$$

Thus, we obtain

$$\int_x^{+\infty}(|\Psi^r_{11}(t,x,y)|^2+|\Psi^r_{12}(t,x,y)|^2)dy \leq C\int_x^{+\infty}(|\mathcal{F}_1(t,x,y)|^2+|\mathcal{F}_2(t,x,y)|^2)dy. (7.15)$$

In particular, if $\mathcal{F}_1(t,x,y)=\mathcal{F}_2(t,x,y)=0$, then

$$\Psi^r_{11}(t,x,y)=\Psi^r_{12}(t,x,y)=0.$$

That is, the homogenous equation has a only trivial solution. Notice that we proved in fact that the above integral equation is coercive, which gives another argument for existence and uniqueness of a solution in $L^2(y \geq x)$. Furthermore, it can be proved that if $\partial^\alpha\{\mathcal{F}_1(t,x,y), \mathcal{F}_1(t,x,y)\} \in L^2_y(y \geq x)$, then there holds

$$\partial^\alpha \Psi^r_{11}(t,x,y), \quad \partial^\alpha \Psi^r_{12}(t,x,y) \in L^2_y(y \geq x).$$

Now, if $\mathcal{F}_1(t,x,y)$ and $\mathcal{F}_2(t,x,y)$ are given by (7.9) and (7.10), it is easily verified by using Proposition 6.3 that for $|\alpha| \leq 2$

$$\|\partial^\alpha \mathcal{F}(t,x,\cdot)\|_{L^2(y\geq x)} \leq C\varepsilon + C\varepsilon \sum_{|\beta|\leq|\alpha|} \|\partial^\beta \Psi(t,x,\cdot)\|_{L^2(y\geq x)}.$$

With this and (7.15), it can be proved by using the fixed point theorem that the integral equations (7.7) and (7.8) has a unique solution $\{\Psi^r_{11}(t,x,y), \Psi^r_{12}(t,x,y) \in L^2_y(y \geq x)\}$ and there hold for $|\alpha| \leq 2$

$$\begin{aligned} &\partial^\alpha \Psi^r_{11}(t,x,y), \quad \partial^\alpha \Psi^r_{12}(t,x,y) \in L^2_y(y \geq x), \\ &\|\partial^\alpha_y \Psi^r_{11}(t,x,\cdot)\|_{L^2(y\geq x)} + \|\partial^\alpha_y \Psi^r_{12}(t,x,\cdot)\|_{L^2(y\geq x)} \leq C\varepsilon. \end{aligned}$$

This completes the proof of the first part of Proposition 7.1.

Next, we show that $u(t,x)$ given by (7.6) is a solution of (1.1). To prove it, we need the following two Lemmas whose proof will be given in the appendix.

Lemma 7.1. *Let*

$$\psi_1(t,x,\lambda) = X^+_1(x,\lambda) - \int_x^{+\infty} \Psi(t,x,s) X^+_1(s,\lambda) ds.$$

Then $\psi_1(t,x,\lambda)$ is the solutions of (2.3) with $q = \frac{\sqrt{2}}{2} u(t,x)$.

Lemma 7.2. *Let $\psi_1(t,x,\lambda)$ be as in Lemma 7.1. Then there holds*

$$\frac{\partial \chi}{\partial t} + iB_u \chi = i\sqrt{3}(\frac{E}{2} - \zeta)^2 \chi, \tag{7.16}$$

with $\chi_1 = (\sqrt{3}-1)^{\frac{1}{2}} e^{i\frac{Ex}{2}} \psi_{11}, \chi_2 = (\sqrt{3}+1)^{\frac{1}{2}} e^{i\frac{Ex}{2}} \psi_{12}, \lambda = \frac{\sqrt{3}}{2} E.$

With Lemma 7.1 and Lemma 7.2, we get by repeating the argument of section 4 that

$$\frac{d}{dt} L_u = i[L_u, B_u],$$

which implies that $u(t,x)$ is a solution of (1.1). □

With Proposition 7.1, we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We recall that we denoted by Ψ^0 the unperturbed kernel given by (2.24). In view of (2.24) and (7.3), it is easy to verify, by using Lemma 3.2, that

$$\|\Psi^{(1)}(t, x + 2\lambda_0 t, y + 2\lambda_0 t) - \Psi^0(x, y)\|_{L^\infty(y \geq x)} \leq C\varepsilon. \tag{7.17}$$

We get by Proposition 7.1 and the Sobolev imbedding that

$$\|\Psi(t, x, y) - \Psi^{(1)}(t, x, y)\|_{L^\infty(y \geq x)} \leq C\varepsilon,$$

from which and (7.17), it follows that

$$\|\Psi(t, x + 2\lambda_0 t, x + 2\lambda_0 t) - \Psi^0(x, x)\|_{L^\infty} \leq C\varepsilon.$$

Note that

$$U_0(x) = \tanh(\frac{x}{\sqrt{2}}) = 2\sqrt{2} i \Psi^0_{21}(x, x) + 1.$$

Thus, we have

$$\|u(t, x + 2\lambda_0 t) - U_0(x)\|_{L^\infty} \leq 2\sqrt{2}\|\Psi(t, x + 2\lambda_0 t, x + 2\lambda_0 t) - \Psi^0(x, x)\|_{L^\infty} \leq C\varepsilon.$$

This completes the proof of Theorem 1.1. □

8 Appendix A

In this Appendix, we will prove Lemma 7.1 and Lemma 7.2.

Proof of Lemma 7.1. To show that $\psi_1(t, x, \lambda)$ is the solution of (2.3), it suffices to prove that $\Psi(t, x, y)$ satisfies the linear system (2.11). For this, we put

$$\begin{aligned} C_1(t, x, y) &= (\partial_x + \partial_y)\Psi_{11}(t, x, y) - i\Big(-\frac{\sqrt{2}}{2}\Psi_{12}(t, x, y) + q^*(t, x)\Psi^*_{12}(t, x, y)\Big), \\ C_2(t, x, y) &= (\partial_x - \partial_y)\Psi_{12}(t, x, y) - i\Big(-\frac{\sqrt{2}}{2}\Psi_{11}(t, x, y) + q^*(t, x)\Psi^*_{11}(t, x, y)\Big). \end{aligned}$$

Then by a direct calculation, we find that $C_1(t, x, y)$ and $C_2(t, x, y)$ satisfy the homogenous Marchenko equations

$$\begin{aligned} 2\sqrt{2}C_1(t, x, y) &= -\int_x^{+\infty} C_2(t, x, s)\sqrt{2}(F_1(t, s + y) - iF_2'(t, s + y)) + C_1(t, x, s)F_2(t, s + y)ds, \\ 2\sqrt{2}C_2(t, x, y) &= -\int_x^{+\infty} C_1(t, x, s)\sqrt{2}(F_1(t, s + y) + iF_2'(t, s + y)) + C_2(t, x, s)F_2(t, s + y)ds. \end{aligned}$$

Since the homogenous equations have a only trivial solution, $C_1(t, x, y) = C_2(t, x, y) = 0$ follows. This proves that $\Psi(t, x, y)$ satisfies the linear system (2.11). □

Proof of Lemma 7.2. Set $h = e^{i\zeta x}\psi_1(t,x,\lambda)$. Then (7.16) is equivalent to

$$\frac{\partial h}{\partial t} = -i\begin{pmatrix} \frac{|u|^2-1}{\sqrt{3}+1} & 0 \\ 0 & \frac{|u|^2-1}{\sqrt{3}-1} \end{pmatrix} h + \begin{pmatrix} 0 & \frac{\sqrt{3}+1}{\sqrt{2}}u_x^* \\ -\frac{\sqrt{3}-1}{\sqrt{2}}u_x & 0 \end{pmatrix} h$$
$$+(2\sqrt{3}\zeta - 2\lambda)\partial_x h + i\sqrt{3}\partial_x^2 h \triangleq g. \tag{8.1}$$

Since $\psi_1(t,x,\lambda)$ is a solution of (2.3), we have

$$\lambda\partial_x h = \begin{pmatrix} 0 & q_x^* \\ q_x & 0 \end{pmatrix} h + \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}\partial_x h + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(i\partial_x^2 h + \zeta\partial_x h).$$

Substituting this into g, we get

$$g = -i\begin{pmatrix} \frac{|u|^2-1}{\sqrt{3}+1} & 0 \\ 0 & \frac{|u|^2-1}{\sqrt{3}-1} \end{pmatrix} h + \begin{pmatrix} 0 & \frac{\sqrt{3}+1}{\sqrt{2}}u_x^* \\ -\frac{\sqrt{3}-1}{\sqrt{2}}u_x & 0 \end{pmatrix} h - 2\begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}\partial_x h$$
$$+2\begin{pmatrix} \sqrt{3}-1 & 0 \\ 0 & \sqrt{3}+1 \end{pmatrix}\zeta\partial_x h + i\begin{pmatrix} \sqrt{3}-2 & 0 \\ 0 & \sqrt{3}+2 \end{pmatrix}\partial_x^2 h. \tag{8.2}$$

Note that

$$h = \begin{pmatrix} 1 \\ \sqrt{2}(\lambda-\zeta) \end{pmatrix} - \int_0^{+\infty} \Psi(t,x,x+s)X_1^+(s,\lambda)ds.$$

Substituting this into (8.2), we get by integrating by parts and (2.11) that

$$g = \int_0^{+\infty} G(t,x,y+s)X_1^+(s,\lambda),$$

where

$$G(t,x,y) = \begin{pmatrix} i(\partial_x+\partial_y)^2\Psi_{11} + 2q^*(\partial_x+\partial_y)\Psi_{21} & i(\partial_x+\partial_y)^2\Psi_{12} + 2q^*(\partial_x+\partial_y)\Psi_{22} \\ -i(\partial_x+\partial_y)^2\Psi_{21} + 2q(\partial_x+\partial_y)\Psi_{11} & -i(\partial_x+\partial_y)^2\Psi_{22} + 2q(\partial_x+\partial_y)\Psi_{12} \end{pmatrix}.$$

Note that

$$\frac{\partial h}{\partial t} = -\int_0^{+\infty} \partial_t\Psi(t,x,x+s)X_1^+(s,\lambda)ds.$$

Thus, to prove (8.1), it suffices to prove that

$$\partial_t\Psi(t,x,y) = -G(t,x,y). \tag{8.3}$$

By the definition of $F_1(t,z)$ and $F_1(t,z)$, we have

$$\frac{\partial F_1}{\partial t}(t,z) = -2\frac{\partial F_2}{\partial z}(t,z) + 2\frac{\partial^3 F_2}{\partial z^3}(t,z),$$
$$\frac{\partial F_2}{\partial t}(t,z) = -4\frac{\partial F_1}{\partial z}(t,z).$$

Thus, we get by differentiating (7.1) and (7.2) with respect to t that

$$
\begin{aligned}
&2\sqrt{2}\partial_t\Psi_{11}(t,x,y) = D_1(t,x,y)\\
&-\int_x^{+\infty}\partial_t\Psi_{12}(t,x,s)\sqrt{2}(F_1(t,s+y)-iF_2'(t,s+y))+\partial_t\Psi_{11}(t,x,s)F_2(t,s+y)ds,\\
&2\sqrt{2}\partial_t\Psi_{12}(t,x,y) = D_2(t,x,y)\\
&-\int_x^{+\infty}\partial_t\Psi_{11}(t,x,s)\sqrt{2}(F_1(t,s+y)+iF_2'(t,s+y))+\partial_t\Psi_{12}(t,x,s)F_2(t,s+y)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1(t,x,y) =& -\int_x^{+\infty}\Psi_{12}(t,x,s)\sqrt{2}(4F_2^{'''}(t,s+y)-2F_2'(t,s+y)+4iF_1^{''}(t,s+y))ds\\
&+4\int_x^{+\infty}\Psi_{11}(t,x,s)F_1'(t,s+y)ds-4F_1'(t,x+y),\\
D_2(t,x,y) =& \sqrt{2}(4F_2^{'''}(t,x+y)-2F_2'(t,x+y)-4iF_1^{''}(t,x+y))\\
&-\int_x^{+\infty}\Psi_{11}(t,x,s)\sqrt{2}(4F_2^{'''}(t,s+y)-2F_2'(t,s+y)-4iF_1^{''}(t,s+y))ds\\
&+4\int_x^{+\infty}\Psi_{12}(t,x,s)F_1'(t,s+y)ds.
\end{aligned}
$$

On the other hand, differentiating (7.1) and (7.2) with respect to x and y, we get by integrating by parts and (2.11) that

$$
\begin{aligned}
&2\sqrt{2}G_{11}(t,x,y) = -D_1(t,x,y)\\
&-\int_x^{+\infty}G_{12}(t,x,s)\sqrt{2}(F_1(t,s+y)-iF_2'(t,s+y))+G_{11}(t,x,s)F_2(t,s+y)ds,\\
&2\sqrt{2}G_{12}(t,x,y) = -D_2(t,x,y)\\
&-\int_x^{+\infty}G_{11}(t,x,s)\sqrt{2}(F_1(t,s+y)+iF_2'(t,s+y))+G_{12}(t,x,s)F_2(t,s+y)ds.
\end{aligned}
$$

Thus, $\partial_t\Psi(t,x,y)$ and $-G(t,x,y)$ satisfy the same Marchenko equations. So, we conclude (8.3) by the uniqueness of the solution of the Marchenko equations. □

9 Appendix B

In this appendix, we prove Corollary 1.1. The proof is based on Theorem 1.1 and a compactness argument.

Proof. Let u_0 be a Cauchy datum as in Theorem 1.1. Due to Theorem 1.1,

$$\sup_{t\in\mathbb{R}}\|u(x+2\lambda_0 t,t)-U_0(x)\|_{L^\infty}\le C\varepsilon\ ,$$

where λ_0 is the unique zero of the transition coefficient a associated to u_0. We shall show that, given, $\delta>0$, for $\varepsilon>0$ small enough,

$$\sup_{t\in\mathbb{R}}\|u'(x+2\lambda_0 t,t)-U_0'(x)\|_{L^2}+\||u(x+2\lambda_0 t,t)|^2-|U_0(x)|^2\|_{L^2}\le\delta.$$

By contradiction, otherwise there would exist $\delta > 0$ and a sequence $\{u_0^n\}$ verifying

$$\| < x >^4 \partial^k (u_0^n(x) - U_0(x))\|_{L^\infty} \to 0, \quad \text{as} \quad n \to +\infty, \qquad \text{for} \quad k \leq 3$$

and a sequence $\{t_n\}$ such that

$$\|u_n'(t_n, x + 2\lambda_0^n t_n) - U_0'(x)\|_{L^2} + \||u_n(t_n, x + 2\lambda_0 t_n)|^2 - |U_0(x)|^2\|_{L^2} \geq \delta \ , \tag{9.1}$$

$$\sup_{t\in\mathbb{R}} \|u_n(t, x + 2\lambda_0 t) - U_0(x)\|_{L^\infty} \to 0, \quad \text{as} \quad n \to +\infty. \tag{9.2}$$

Here λ_0^n denotes the unique zero of the transition coefficient a_n associated to u_0^n, and u_n denotes the solution to (1.1) with Cauchy datum u_0^n. We will follow the argument in Page 20 of [4] to yield a contradiction. Firstly, we have the energy conservation

$$E(u_n(t)) = \frac{1}{2}\int |u_n'(t,x)|^2 dx + \frac{1}{4}\int ||u_n(t,x)|^2 - 1|^2 dx = E(u_0^n),$$

from which and (9.2), it follows that,for any $B > 0$,

$$u_n(t_n, x + 2\lambda_0^n t_n) \rightharpoonup U_0(x) \quad \text{in} \quad H^1(|x| \leq B). \tag{9.3}$$

Next, we take B such that

$$\int_{|x|\geq B} \frac{1}{2}|U_0'|^2 + \frac{1}{4}||U_0|^2 - 1|^2 dx \leq \frac{\delta}{4}, \tag{9.4}$$

and n_δ such that

$$E(u_0^n) \leq E(U_0) + \frac{\delta}{4}, \tag{9.5}$$

for any $n \geq n_\delta$. Since

$$\int_{|x|\leq B} \frac{1}{2}|U_0'|^2 + \frac{1}{4}||U_0|^2 - 1|^2 dx \leq \liminf_{n\to+\infty} \int_{-B+2\lambda_0^n t_n}^{B+2\lambda_0 t_n} \frac{1}{2}|u_n'(t_n,x)|^2 + \frac{1}{4}\|u_n(t_n,x)|^2 - 1|^2 dx$$

which together with (9.4)-(9.5) and the energy conservation implies that for $n \geq n_\delta$

$$\int_{|x-2\lambda_0^n t_n|\geq B} \frac{1}{2}|u_n'(t_n,x)|^2 + \frac{1}{4}\|u_n(t_n,x)|^2 - 1|^2 dx \leq \frac{\delta}{2},$$

In view of (9.3), we infer, for n big enough

$$\|u_n'(t_n, x + 2\lambda_0^n t_n) - U_0'(x)\|_{L^2} + \||u_n(t_n, x + 2\lambda_0 t_n)|^2 - |U_0(x)|^2\|_{L^2} < \delta$$

which contradicts (9.1). The corollary follows. $\square$

References

[1] F. Béthuel and J.-C. Saut, Travelling waves for the Gross-Pitaevskii equation I. Ann. Inst. H. Poincaré Phys. Théor., 70 (1999), 147–238.

[2] F. Béthuel, G. Orlandi and D. Smets, Vortex rings for the Gross-Pitaevskii equation. J. Eur. Math. Soc., 6 (2004), 17–94.

[3] F. Béthuel, P. Gravejat and J.-C. Saut, Existence and properties of traveling waves for the Gross-Pitaevskii equation. To appear in *Stationary and Time Dependent Gross-Pitaevskii Equations*, A. Farina and J.-C. Saut editors, Contemporary Mathematics, American Mathematical Society, 2008.

[4] F. Béthuel, P. Gravejat, J.-C. Saut and D. Smets, Orbital stability of the black soliton to the Gross-Pitaevskii equation. To appear in Indiana Math. J., special volume dedicated to C. Foias.

[5] L. Di Menza and C. Gallo, The black solitons of one-dimensional NLS equations. Nonlinearity, 20(2007), 461–496.

[6] A. Farina and J.-C. Saut, *Stationary and Time Dependent Gross-Pitaevskii Equations*, Contemporary Mathematics, American Mathematical Society, 2008.

[7] P. Gérard, The Cauchy problem for the Gross-Pitaevskii equation. Ann. Inst. H. Poincaré Anal. Non Linéaire, 23(2006), 765–779.

[8] P. Gérard, The Gross-Pitaevskii equation in energy space, to appear in *Stationary and Time Dependent Gross-Pitaevskii Equations*, A. Farina and J.-C. Saut editors, Contemporary Mathematics, American Mathematical Society, 2008.

[9] M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry I. Jour. Funct. Anal., 74(1987), 160–197.

[10] Z. Lin, Stability and instability of traveling solitonic bubbles. Adv. Differential Equations, 7(2002), 897–918.

[11] V. E. Zakharov and A. B. Shabat, Interaction between solitons in a stable medium. Sov. Phys. JETP, 37(1973), 823-828.

[12] P. E. Zhidkov, Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory. Lecture Notes in Mathematics 1756, Springer-Verlag, Berlin, 2001.